- Class website

  mathweb.ucsd.edu/~lni/math240

- All information can be found
- Reading: means you need to read.
- My lectures will try to center on some important results.

Lecture notes shall be sent afterwards to Canvas & posted at the website.

Grades: 40% + 30% + 30%

- Functional Analysis
- L^p-spaces
- Some weak topology, Compactness,
- Radon measures. — Continue some topics on measures left by Peter Ebenfelt in Ch 3.

1st Lecture is on Hahn-Banach Theorem.

In the course announcement via Canvas I have assigned the reading of §4.1 & §4.2. Please do read them and do the exercises since we do assume some basics from the general topology.

Hahn-Banach Theorem & Applications

① Readings:

- Normed Vector spaces & a semi-norm
- Banach spaces Thm 5.1
- Products & quotients
- Bounded linear map
- Banach algebra

② **Thm 1** (Hahn-Banach) Let $\mathcal{X}$ be a real vector space, p a sublinear functional, $\mathcal{M}$ a subspace of $\mathcal{X}$.
Let f be a linear functional on $\mathcal{M}$ such that $f(x) \leq p(x)$ $\forall x \in \mathcal{M}$.
Then $\exists$ a linear functional F on $\mathcal{X}$ with ⓐ $F(x) \leq p(x)$ $\forall x \in \mathcal{X}$
ⓑ $F|_{\mathcal{M}} = f$.

- Linear function $f: \mathcal{X} \to \mathbb{R}$, $\begin{cases} f(x+y) = f(x) + f(y) \\ f(\lambda x) = \lambda f(x) \end{cases}$
- Sub- $p: \mathcal{X} \to \mathbb{R}$ $p(x+y) \leq p(x) + p(y)$, $p(\lambda x) = \lambda p(x)$ $\forall \lambda \geq 0$

Hahn-Banach: An extension with estimates.

③ — proof — step 1, — finite dimensional case (Namely $\mathcal{X} = \mathcal{M} +$ linear span of $\{x_1 \dots x_\ell\}$)

$\mathcal{M} + \mathbb{R}\{x\}$ case.

$$F(y + \lambda x) = f(y) + \lambda \alpha$$

Pick an α such that it satisfies the estimate:

$$F(y+\lambda x) \leq p(y+\lambda x)$$
$$\| $$
$$f(y) + \lambda\alpha \leq p(y+\lambda x)$$

Case 1: $\lambda > 0$ $\quad \alpha \leq -\frac{1}{\lambda} f(y) + \frac{1}{\lambda} p(y+\lambda x)$

$$= -f(\tfrac{1}{\lambda} y) + p(\tfrac{1}{\lambda} y + x)$$

Case 2: $\lambda < 0$ $\quad \alpha \geq \frac{1}{\lambda}\left(p(y+\lambda x) - f(y)\right) = -p(-\frac{1}{\lambda} y - x) + f(-\frac{y}{\lambda})$

$$\Longleftrightarrow \begin{cases} \alpha \leq P(x+y') - f(y') \\ \alpha \geq f(y'') - P(y''-x) \end{cases}$$

This however can be checked, namely

$$P(x+y') - f(y') \geq f(y'') - P(y''-x)$$

$$\Longleftrightarrow \quad P(x+y') + P(y''-x) \geq f(y') + f(y'') = f(y'+y'').$$

4 Zorn's Lemma: If X is a partially ordered set, such that every linearly ordered subset of X has an upper bound, then X has a maximal element.

Partial order, 1 $x \leq y \quad y \leq z \Rightarrow x \leq z$

2 if $x \leq y$ & $y \leq x \Rightarrow x = y$

3 $x \leq x$

Linear order on S. $\forall x, y \in S$. either $x \leq y$ or $y \leq x$.

5 Application to finish the proof of Hahn-Banach.

$$X := \left\{ (Z, G) \;\middle|\; \begin{matrix} M \subset Z & \& \quad G \leq P \text{ on } Z \\ G|_M = f & \end{matrix} \right\}$$

$(Z, G) \leq (Z', G')$

if $Z \subset Z'$

& $G'|_G = G$

$\{(Z_\nu, G_\nu)\}$ a linearly ordered subset, $\begin{cases} Z := \bigcup Z_\nu \\ G|_{Z_\nu} := G_\nu \end{cases}$

$\Rightarrow \quad (Z_\gamma, G_\gamma) \leq (Z, G)$

Hence Zorn's Lemma assumption is Satisfied.

$\Rightarrow$ $\exists$ a maximal element. $\Rightarrow$ F exists on X (otherwise it is NOT maximal by ③

⑥ Applications:

Theorem 5.8: a. Seperation of M & x: $x \notin M$. $\exists f$ $f|_M = 0$. $f(x) = \delta$. $\|f\| = 1$. $\delta = \inf_{y \in M} \|x - y\|$.

b. If $x \neq 0 \in X$. $\exists f \in X^*$ $\|f\| = 1$ & $f(x) = \|x\|$

c. $\forall x \neq y \in X$ $\exists$ $f \in X^*$ $f(x) \neq f(y)$

d. $\forall x \in X$ $\hat{x} := f(x)$ $\forall f \in X^*$
Then $x \to \hat{x}$: $X \to X^{**}$ is a linear isometry.

Pf: a. $f(y + \lambda x) \doteq \lambda \delta$

Then f is a linear functional on $M + \mathbb{R}\{x\}$

$$\left| f(y + \lambda x) \right| = |\lambda| \delta \leq |\lambda| \left\| x + \tfrac{1}{\lambda} y \right\| = \| \lambda x + y \|$$

Hence $|f(z)| \leq |z|$ on $M' := M + \mathbb{R}\{x\}$

Now apply HB to M'

$\Rightarrow \quad |F(z)| \leq |z| \quad \forall z \in X \quad \Rightarrow \quad \|F\| \leq 1$

But $|F(x)| = \delta$ $\quad \exists$ $y_n \in M$ $\|x - y_n\| \to \delta$

$F(x - y_n) = \delta \Rightarrow |F(x - y_n)| = \delta$

Hence $\|F\| \geq \dfrac{|F(x - y_n)|}{\|x - y_n\|} \to 1$

For d. $\hat{x}$ is clearly linear

$\hat{x}(f) = 0 \ \forall f \Rightarrow x = 0$ otherwise $\exists f \in X^*$ $f(x) = \|x\| \neq 0$

$\Rightarrow \hat{x}$ is 1-1.

& $|\hat{x}(f)| = |f(x)| \leq \|f\| \|x\| \Rightarrow \|\hat{x}\| \leq \|x\|$

On the other hand by b. $\Rightarrow \exists f_0$, $\|f_0\| = 1$, $\hat{x}(f_0) = f_0(x) = \|x\|$

$\Rightarrow \dfrac{|\hat{x}(f_0)|}{\|f_0\|} = \|x\| \Rightarrow \|\hat{x}\| \geq \|x\|$.

• Reading: • Reflexive normed spaces.

• Complex Hahn-Banach Theorem

① Prop 5.5 & Thm 5.7 of FD

② Further applications §3.2 of LAX.