Math 240, **Final exam** **March, 2015**
Ni @ UCSD

Name: __________________________

Student #: __________________________

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General instructions: 3 hours. Be sure to carefully motivate all (nontrivial) claims and statements. You may use without proof any result proved in the text (as well as ones covered in the lecture). If you use a theorem from the text (or lecture), refer to it either by name (if it has one) or explain what it says. Also verify explicitly all hypotheses in the theorem. You need to reprove any result given as an exercise. If the statement you are asked to prove is exactly a result covered in the class you are asked to re-construct the proof instead of citing the result.

1. Short proofs. Here please provide a short, but complete proof of each statement below.

   a) Let X be a reflexive normed vector space. Prove that X is a Banach space.

   Solution: $X = (X^*)^*$. But Y^* is Banach by Proposition 5.4 for any normed vector space Y. Hence X is a Banach space.

   b) Give an example of $L^\infty(X, \mu)$ such that it is not separable with respect to $\|\cdot\|_\infty$. Justify your claim.

   Solution: $L^\infty([0,1])$ is an example. Define $f_t = \chi_{[0,t]}$. Clearly $\|f_t - f_{t'}\|_\infty = 1$ for any $t \neq t'$. Hence $B(f_t, \frac{1}{3})$ are disjoint open balls.

   Then if that $L^\infty([0,1])$ is separable, there exists countable dense subset $\{g_n\}$. But each ball $B(f_t, \frac{1}{3})$ has to contain some g_{n_t} since $\{g_n\}$ is dense. This gives $t \to n_t$ an injective map from $[0,1]$ into N, the natural numbers. This is absurd.

c) Prove that $(L^\infty[0,1])^* \neq L^1[0,1]$.

Solution: See page 191 of the book. You do not need to use the last problem and problem 2 to solve this. It can be done by direct construction.

d) Let X be a locally compact Hausdorff space. Prove that the Dirac measure $\delta_{x_0}(y)$ (which is centered at some $x_0 \in X$) is a Radon measure.

Solution: Check 1) Δ_{x_0} is a measure on Borel algebra.

2) Clearly $\delta_{x_0}(K) = 1$ if $x_0 \in K$ or $\delta_{x_0}(K) = 0$ otherwise, hence $< \infty$.

3) Check inner regular on open set. If $x_0 \in U$ then the result follows by the existence of $x_0 \in K \subset U$. Otherwise $\delta_{x_0}(U) = 0$, $\delta_{x_0}(K) = 0$ for any $K \subset U$.

4) Outer regular on Borel E. If $x_0 \in E$ then $\delta_{x_0}(E) = 1$ and $\delta_{x_0}(U) = 1$ for any $E \subset U$. If x_0 does not belong to E, then $U = X \subset \{x_0\}$ is open and $\delta_{x_0}(U) = 0$, $\delta_{x_0}(E) = 0$.

e) Assume that the normed space X is separable. Then for any sequence $\{f_n\}$ in a bounded subset of X^*, there exists a subsequence $\{f_{n_k}\}$ and $f \in X^*$ such that $f_{n_k}(x) \to f(x)$ for any $x \in X$.

solution: Pick a countable dense subset $\{x_j\} \subset X$. By diagonal process there exists subsequence f_{n_k} such that $\{f_{n_k}(x_j)$ convergent. Define $f(x_j) = \lim f_{n_k}(x_j)$ and then define $f(y)$ by let $f(y) = \lim f(x_{j_k})$ for the sequence $x_{j_k} \to y$. One can check that this is well defined.

Check f is linear.

Check that $f_{n_k}(y) \to f(y)$ so defined.

You do not use Alaoglu, which involves many other statements/exercises.

f) Let X be a locally compact Hausdorff space. Assume that μ is a Radon measure. Let $\phi \geq 0$ be a integrable function defined on X. Define the measure ν by

$$\nu(E) = \int_E \phi(x)\, d\mu(x).$$

Prove that ν is a Radon measure.

Solution: Homework problem 7.8. See the solution to the homework.

2. Let X be a Banach space. Prove that if X^* is separable, then X is separable.

   Solution: Let $\{f_n\}$ be the countable dense subset in X^*. Pick x_n such that $\|x_n\| = 1$ and $|f_n(x_n)| \geq \frac{1}{2}\|f_n\|$.

   Now we show that $Span\{x_1, x_2, \cdots, x_n, \cdots\}$ is dense in X. This is enough since $\{\sum_k a_k x_{n_k}\}$ with a_k having rational real and imaginary parts is countable.

   Let $M = \overline{Span\{x_1, x_2, \cdots, x_n, \cdots\}}$. We shall show $M = X$, otherwise by the Theorem 5.8, there exists $f = 0$ on M but $\|f\| = 1$.

   Then there exists sequence f_{n_k} such that $f_{n_k} \to f$. Now

$$|f_{n_k}(x_{n_k}) - f(x_{n_k})| = |f_{n_k}(x_{n_k})| \geq \frac{1}{2}\|f_{n_k}\|$$

   and

$$|f_{n_k}(x_{n_k}) - f(x_{n_k})| \leq \|f_{n_k} - f\| \to 0$$

   This implies that

$$1 = \|f\| = \lim \|f_{n_k}\| \leq 2|f_{n_k}(x_{n_k}) - f(x_{n_k})| \leq \|f_{n_k} - f\| \to 0.$$

   A contradiction!

3. Let Ω be a bounded domain in $\mathbb{R}^n$. Assume that $f_n \in L^p(\Omega)$ for $1 < p < \infty$ and converges weakly to $f \in L^p(\Omega)$. Show that

   a) $\{f_n\}$ is bounded in $L^p(\Omega)$;

   b) $\|f\|_p \leq \liminf_{n\to\infty} \|f_n\|_p$.

   Solution: See the solution to the practicing final.

4. Assume that X, Y, Z are Banach spaces. Assume $\Phi : X \times Y \to Z$ is bilinear, namely for every $x \in X$, $\Phi(x, \cdot) : Y \to Z$ is linear and for every $y \in Y$, $\Phi(\cdot, y) : X \to Z$ is linear. Show that if for every $z^* \in Z^*$ $z^*(\Phi(x, \cdot)) \in Y^*$ and $z^*(\Phi(\cdot, y)) \in X^*$. then there exists M such that

$$\|\Phi(x, y)\| \leq M\|x\|\|y\|.$$

Solution: See the solution to the practicing final.

5. Assume that $1 \le p < \infty$ and $(X, \mathcal{M}, \mu)$ is a measure space. If $f_n \to f$ in measure and $|f_n| \le g \in L^p(X, d\mu)$ for all n. Then $f_n \to f$ in L^p-norm.

   Solution: See the solution to the practice final

END OF EXAM