Lecture 23: Gauss' Theorem or The divergence theorem. states that if W is a volume bounded by a surface S with outward unit normal $\mathbf{n}$ and $\mathbf{F} = F_1\mathbf{i} + F_2\mathbf{j} + F_3\mathbf{k}$ is a continuously differentiable vector field in W then

$$\iint_S \mathbf{F}\cdot\mathbf{n}\,dS = \iiint_W \operatorname{div}\mathbf{F}\,dV, \qquad \text{where} \quad \operatorname{div}\mathbf{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}.$$

Let us however first look at a one dimensional and a two dimensional analogue. A one dimensional analogue if the First Fundamental Theorem of Calculus:

$$f(b) - f(a) = \int_a^b f'(x)\,dx.$$

A two dimensional analogue says that if D is a region in the plane with boundary curve C and $\mathbf{n} = (n_1, n_2)$ is the outward unit normal to C, then

$$\int_C F_1 n_1 + F_2 n_2\,ds = \iint_D \Big(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y}\Big)\,dA$$

where ds is the arclength. (This is in fact equivalent to Green's Theorem.)

Ex. Find flux of $\mathbf{F} = 2x\,\mathbf{i} + y^2\,\mathbf{j} + z^2\,\mathbf{k}$ out of the unit sphere S.

Sol. By the divergence theorem we have with B the unit ball

$$\iint_S \mathbf{F}\cdot\mathbf{n}\,dS = \iiint_B \operatorname{div} F\,dV \;= \iiint_B (2 + 2y + 2z)\,dV$$
$$= \iiint_B 2dV + \iiint_B 2ydV + \iiint_B 2zdV = 2\operatorname{Vol}(B) + 0 + 0 = 2\frac{4\pi}{3}$$

since the last two integrals vanishes because the region is symmetric under replacing y by $-y$ (respectively z by $-z$) but the integrand changes sign.

Ex. Find flux of $\mathbf{F} = x\,\mathbf{i} + y\,\mathbf{j} - 2z\,\mathbf{k}$ out of the unit sphere.

Sol. By the divergence theorem the flux is equal to the integral of the divergence over the unit ball. Since $\operatorname{div}\mathbf{F} = 0$ it follows that the volume integral vanishes and by the divergence theorem the flux therefore vanishes.

Let B_ε be a ball of radius ε and let S_ε be its surface. Then

$$\iint_{S_\varepsilon} \mathbf{F}\cdot\mathbf{n}\,dS = \iiint_{B_\varepsilon} \operatorname{div}\mathbf{F}\,dV$$

By the mean value theorem for integrals the right hand side is equal to the volume of the box B_ε times $\operatorname{div}\mathbf{F}$ at some point in the box so we get the interpretation of the divergence that we announced in section 3.3:

$$\text{Flux of } \mathbf{F} \text{ out through } S_\varepsilon = \operatorname{Vol}(B_\varepsilon)\operatorname{div}\mathbf{F}$$

where $\operatorname{div}\mathbf{F}$ is evaluated at some point in B_ε. Hence

$$\operatorname{div}\mathbf{F} = \lim_{\varepsilon\to 0} \frac{\text{Flux of } \mathbf{F} \text{ out through } S_\varepsilon}{\operatorname{Vol}(B_\varepsilon)}$$

Proof of the divergence theorem for convex sets.
We say that a domain V is **convex** if for every two points in V the line segment between the two points is also in V, e.g. any sphere or rectangular box is convex.

We will prove the divergence theorem for convex domains V. Since $\mathbf{F} = F_1\mathbf{i} + F_3\mathbf{j} + F_3\mathbf{k}$ the theorem follows from proving the theorem for each of the three vector fields $F_1\mathbf{i}$, $F_2\mathbf{j}$ and $F_3\mathbf{k}$ separately. The theorem for the vector field $F_3\mathbf{k}$ states that

$$\iint_S (F_3\mathbf{k})\cdot\mathbf{n}\,dS = \iiint_D \frac{\partial F_3}{\partial z}\,dV$$

Since V is convex we can write $V = \{(x,y,z);\ f_1(x,y) \le z \le f_2(x,y),\ (x,y)\in D\}$.

Then S consists of two parts $S_1 = \{(x,y,z);\ z = f_1(x,y),\ (x,y)\in D\}$ and $S_2 = \{(x,y,z);\ z = f_2(x,y),\ (x,y)\in D\}$. We have

$$\begin{aligned}\iiint_V \frac{\partial F_3}{\partial z}\,dzdxdy &= \iint_D \int_{f_1(x,y)}^{f_2(x,y)} \frac{\partial F_3}{\partial z}\,dz\ dxdy\\ &= \iint_D F_3\big((x,y,f_2(x,y)\big)\,dxdy - \iint_D F_3\big((x,y,f_1(x,y)\big)\,dxdy\end{aligned}$$

We know that $dxdy = \mathbf{k}\cdot\mathbf{n}\,dS$ on S_2 and $dxdy = -\mathbf{k}\cdot\mathbf{n}\,dS$ on S_1 so (5.1.4) follows.

Ex. Gauss law Let $\mathbf{F} = -(x\mathbf{i}+y\mathbf{j}+z\mathbf{k})/(x^2+y^2+z^2)^{3/2}$. Let S_a be the sphere of radius a centered at the origin. Find the flux of $\mathbf{F}$ out of S_a

Sol. A calculation shows that $\operatorname{div}\mathbf{F} = ... = 0$, when $|x| \neq 0$. Hence by the divergence theorem the flux is $\iint_{S_a}\mathbf{F}\cdot\mathbf{n}\,dS = \iiint_{B_a}\operatorname{div}\mathbf{F}\,dV$, where B_a is the ball of radius a centered at 0. From this we deduce that the flux is 0 but this answer is wrong! In fact the outward unit normal to $S_a = \{(x,y,z);\ x^2+y^2+z^2 = a^2\}$ is $\mathbf{n} = (x\mathbf{i}+y\mathbf{j}+z\mathbf{k})/(x^2+y^2+z^2)^{1/2}$. It therefore follows that

$$\iint_{S_a}\mathbf{F}\cdot\mathbf{n}\,dS = \iint_{S_a}\frac{dS}{x^2+y^2+z^2} = \iint_{S_a}\frac{1}{a^2}\,dS = \frac{1}{a^2}\text{Area}\,(S_a) = \frac{1}{a^2}4\pi a^2 = 4\pi$$

Therefore there appears to be a contradiction and we conclude that the divergence theorem is not valid in this case. In fact $\mathbf{F}$ to be continuously differentiable and bounded in $\overline{V}$ and $\mathbf{F}$ is unbounded at the origin. We also remark that the flux out of any region of $\mathbf{F}$ is 4π if the region contains the origin and 0 if the region does not contain the origin.