Math 140A Test 2 **100 points** **November 23, 2009** **Professor Evans**

Directions: Show all work. In your proofs, state where you use the hypotheses.
Notation: Throughout, z and a_n $(n = 0, 1, 2, 3, \dots)$ are complex. (If you can only handle the real case, you still get considerable partial credit.)
Points: Each problem is worth 20 points.

(1) Given that $a_n \to 0$, show that $(a_1 + \cdots + a_n)/n \to 0$, as $n \to \infty$.

Solution: Let $\epsilon > 0$. Let $s_n = a_1 + \cdots + a_n$. There exists an integer N such that $|a_n| < \epsilon$ for all $n > N$. There exists a positive constant M (independent of ϵ) for which $|a_n| < M$ for all $n \leq N$. Thus by the triangle inequality, $|s_n| \leq NM + (n - N)\epsilon$. Thus $|s_n|/n < NM/n + \epsilon$, so $|s_n|/n < 2\epsilon$ for all $n > NM/\epsilon$. This shows the desired result that $s_n/n \to 0$.

(2) Consider the power series $\sum_{n=0}^{\infty} z^n/(n+1)$. Determine the values of z for which the series **(A)** diverges **(B)** converges. Justify briefly.

Solution: It's easy to see using the ratio test or the root test that the series diverges for $|z| > 1$ and converges (absolutely) for $|z| < 1$. (The ROC is 1.) But what happens when $|z| = 1$? In that case, the series converges by Dirichlet's test (Theorem 3.42), unless $z = 1$, in which case this series is the divergent harmonic series.

(3) Given that $|z| < 1$, prove that $\sum_{n=0}^{\infty} z^n = 1/(1-z)$. Justify every step in the proof.

Solution: $\sum_0^T z^n = (1-z^{T+1})/(1-z)$, since z is not equal to 1. This is proved by cross-multiplying and canceling. To find the value of the infinite series, take the limit as $T \to \infty$. Since $|z| < 1$, the right side approaches the desired limit $(1-0)/(1-z) = 1/(1-z)$.

(4) Let $\{a_n\}$ be a monotone increasing sequence of reals with least upper bound 5. Prove that $a_n \to 5$ as $n \to \infty$.

Solution: Let $\epsilon > 0$. For some N, a_N lies in the interval $[5-\epsilon, 5]$, otherwise 5 would not be the least upper bound of the sequence. Since the sequence is monotone increasing, it follows that a_n lies in the interval $[5-\epsilon, 5]$ for all $n > N$. By definition of a limit of a sequence, it follows that $a_n \to 5$.

(5) **(A)** Give an example of a sequence $\{a_n\}$ for which $\sum_{n=1}^{\infty} a_n$ converges but $\sum_{n=1}^{\infty} a_n^2$ diverges. Justify briefly.
(B) Show that if $\sum_{n=1}^{\infty} a_n$ converges absolutely, then $\sum_{n=1}^{\infty} a_n^2$ converges absolutely as well.

Solution: **(A)** $a_n = (-1)^n/\sqrt{n}$ works, by Dirichlet's test and the divergence of the harmonic series.
Solution: **(B)** Since $\sum_{n=1}^{\infty} |a_n|$ converges, $a_n \to 0$. Thus there is an integer N for which $|a_n| < 1$ for all $n \geq N$. Therefore $|a_n|^2 < |a_n|$ for all $n \geq N$. Thus by the comparison test, $\sum_{n=N}^{\infty} |a_n|^2$ converges. Thus $\sum_{n=1}^{\infty} |a_n|^2$ converges.