Problem set 6

Do for Friday, May 11.

1. Prove that the family $\{\sin\ nx, n = 1, 2, 3 \ldots\}$ is not equicontinuous on the interval $[-1, 1]$.

2. Prove that the family of all polynomials of degree $\leq N$ with coefficients in the interval $[-1, 1]$ is uniformly bounded and equicontinuous on any compact interval.

3. For any continuous, real valued function f on $[0, 1]$, let $F_f(x) = \int_0^x f(t)\ dt$. Show that the set of functions

$$\mathcal{F} = \{F_f : ||f|| \leq 1\}$$

is bounded and equicontinuous.

4. Give an example of a metric space X and a sequence of functions $\{f_n\}$ on X such that $\{f_n\}$ is equicontinuous but not uniformly bounded.

5. Give an example of a uniformly bounded and equicontinuous sequence of functions on $\mathbb{R}$ which does not have any uniformly convergent subsequences.

6. Let X be a metric space such that $X = \bigcup_{n=1}^{\infty} K_n$, where each K_n is compact and such that any bounded open set U is contained in K_n for some n. (An example is $X := \mathbb{R}^k$ with $K_n := \{x \in \mathbb{R}^n : \|x\| \leq n\}$.) Let $\{f_j\}$ be a sequence of functions which are pointwise bounded on X and whose restriction to any K_n is equicontinuous. Show that there exists a subsequence $\{f_{n_j}\}$ that converges to a continuous function on X.

Hint: Use a diagonal trick.