Problem set 4

Do for Friday, April 27

1. Suppose f is a real, continuously differentiable function on [a,b] with f(a)=f(b)=0, and

$$\int_a^b f^2(x)\ dx = 1.$$

Show that

$$\int_a^b xf(x)f'(x)\ dx = -1/2.$$

2. Prove that if $f \in \mathcal{R}[a,b]$ and g is a function for which $g(x) = f(x)$ for all x except for a finite number of points, then g is Riemann integrable. Is the result still true if $g(x) = f(x)$ for all x except for a countable number of points?

3. Let $f : [0,\infty) \to \mathbb{R}$ be defined as $f(x) = 0$ if $0 \leq x \leq 1/2$ and $f(x) = 1$ if $1/2 < x \leq \infty$. Show that the function

$$F(x) = \int_0^x f(t)dt,$$

defined for $0 \leq x < \infty$, is differentiable for $x \neq 1/2$ and is not differentiable for $x = 1/2$.

4. Prove that if f and g are Riemann integrable on $[a,b]$ (i.e. $f, g \in \mathcal{R}[a,b]$) and there exists $N > 0$ such that $g(x) \geq 1/N$ for all $x \in [a,b]$, then $f/g \in \mathcal{R}[a,b]$. (Hint: First show $1/g \in \mathcal{R}[a,b]$, and then use Theorem 6.13.)