MATH 140A - HW 6 SOLUTIONS

Problem 1 (WR Ch 3 #21)**.** Prove the following analogue of Theorem 3.10(b): If $\{E_n\}$ is a sequence of closed nonempty and bounded sets in a complete metric space X, if $E_n \supset E_{n+1}$, and if

$$\lim_{n\to\infty} \operatorname{diam} E_n = 0,$$

then $\bigcap_1^\infty E_n$ consists of exactly one point.

Solution. If $\bigcap E_n$ has two or more points, say $x, y \in \bigcap E_n$ with $x \neq y$, then $x, y \in E_n$ for all n, so

$$\operatorname{diam} E_n \geq d(x, y) > 0,$$

contradicting the fact that $\operatorname{diam} E_n \to 0$. Therefore, we know it has at most one point; all that is left is to prove it is nonempty.

For each n, pick a point x_n in E_n (which is nonempty). Since $E_n \supset E_{n+1}$, we know $\{x_n, x_{n+1}, \ldots\} \subset E_n$. The fact that $\operatorname{diam} E_n \to 0$ implies that for any $\epsilon > 0$ we can pick some $N \in \mathbb{N}$ such that $\operatorname{diam} E_n < \epsilon$ for $n \geq N$, which means for any $n, m \geq N$ we have

$$|x_n - x_m| \leq \operatorname{diam}(\{x_N, x_{N+1}, x_{N+2}, \ldots\}) \leq \operatorname{diam} E_N < \epsilon,$$

so $\{x_n\}$ is a Cauchy sequence. Since X is complete, that means that $\{x_n\}$ converges, say $x_n \to x$. E_n is closed and it contains $\{x_n, x_{n+1}, \ldots\}$, (i.e. the tail of the sequence $\{x_n\}$) so $x \in E_n$ for all n. Therefore, $x \in \bigcap E_n$, so it is nonempty, finishing the proof.

Problem 2 (WR Ch 3 #22)**.** Suppose X is a nonempty complete metric space, and $\{G_n\}$ is a sequence of dense open subsets of X. Prove Baire's theorem, namely, that $\bigcap_1^\infty G_n$ is not empty. (In fact, it is dense in X.)

Solution. Let $A = \bigcap_1^\infty G_n$. We will prove A is dense, which will show it is nonempty (since if $A = \emptyset$ and A is dense then $X = \overline{A} = \emptyset$, contradicting the fact that X is nonempty). To do that, we'll first prove the following claim:

Claim. $A \subset X$ is dense iff $N \cap A \neq \emptyset$ for every neighborhood $N \subset X$.

Proof of claim.

$\boxed{\Longrightarrow}$ Assuming $A \subset X$ is dense, then $A \cup A' = \overline{A} = X$. Then for any $x \in N$, there are two possible cases:

- **(a)** $x \in A$, in which case $x \in N \cap A$, so $N \cap A \neq \emptyset$.
- **(b)** $x \notin A$ and $x \in A'$. Then since x is a limit point of A, and N is a neighborhood of x, there is some $y \in N$ such that $y \neq x$ and $y \in A$. But then $y \in N \cap A$, so $N \cap A \neq \emptyset$.

$\boxed{\Longleftarrow}$ Assuming $N \cap A \neq \emptyset$ for every neighborhood $N \subset X$, we want to show that $\overline{A} = X$. Let $x \in X$. If $x \in A$ we are done. Otherwise, assume $x \notin A$, and we must show that x is a limit point of A. For any neighborhood N of x, we know that $N \cap A \neq \emptyset$, so there must be some $y \in N \cap A$. However, since $x \notin A$, that means that $y \neq x$. Therefore x is a limit point of A, finishing the proof of the claim.

Now, using the claim, let N_0 be any neighborhood in X, and we'll show that $N_0 \cap A \neq \emptyset$. Since G_1 is a dense open subset of X, we know that $N \cap G_1 \neq \emptyset$, so take any $x_1 \in N_0 \cap G_1$. Since $N_0 \cap G_1$ is open, there is a neighborhood N_1 of x_1, and we can shrink N_1 if necessary to guarantee that $\operatorname{diam} \overline{N_1} \leq 1$. Let $E_1 = \overline{N_1}$. We repeat this process, so that if x_{n-1}, N_{n-1} and E_{n-1} are already defined, we define the next terms in the sequence as follows: since G_n is a dense open subset of X, we know that $N_{n-1} \cap G_n \neq \emptyset$, so take any $x_n \in N_{n-1} \cap G_n$. Since $N_{n-1} \cap G_n$ is open, there is a neighborhood N_n of x_n, and we can shrink N_n if necessary to guarantee that $\operatorname{diam} \overline{N_n} \leq \frac{1}{n}$. Let $E_n = \overline{N_n}$. Then $\{E_n\}$ is a collection satisfying the conditions of the previous exercise, so $\bigcap E_n = \{p\}$ for some point $p \in N_0$, and since $E_n \subset G_n$, that means there is a point p in each G_n, and thus $p \in A = \bigcap G_n$. Therefore, $N_0 \cap A \neq \emptyset$.

Problem 3 (WR Ch 3 #23). Suppose $\{p_n\}$ and $\{q_n\}$ are Cauchy sequences in a metric space X. Show that the sequence $\{d(p_n, q_n)\}$ converges.

Solution. For any m, n,

$$d(p_n, q_n) \leq d(p_n, p_m) + d(p_m, q_m) + d(q_m, q_n) \qquad \Longrightarrow \qquad d(p_n, q_n) - d(p_m, q_m) \leq d(p_n, p_m) + d(q_m, q_n).$$

Doing the same argument but switching m and n, we also have

$$d(p_m, q_m) - d(p_n, q_n) \leq d(p_m, p_n) + d(q_n, q_m),$$

so putting these two together we have

$$|d(p_n, q_n) - d(p_m, q_m)| \leq d(p_n, p_m) + d(q_m, q_n).$$

For any $\epsilon > 0$, since $\{p_n\}$ is Cauchy, we can find an $N_1 \in \mathbb{N}$ such that for $m, n \geq N_1$, $d(p_n, p_m) < \epsilon/2$, and since $\{q_n\}$ is Cauchy, we can find an $N_2 \in \mathbb{N}$ such that for $m, n \geq N_2$, $d(q_n, q_m) < \epsilon/2$. Therefore, for $n, m \geq N = \max(N_1, N_2)$ we have

$$|d(p_n, q_n) - d(p_m, q_m)| \leq d(p_n, p_m) + d(q_m, q_n) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon,$$

so $\{d(p_n, q_n)\}$ is a Cauchy sequence, and since $\mathbb{R}$ is a complete metric space and $\{d(p_n, q_n)\}$ is a Cauchy sequence of real numbers, it converges.

Problem 4 (WR Ch 4 #1). Suppose f is a real function defined on $\mathbb{R}^1$ which satisfies

$$\lim_{h \to 0} [f(x+h) - f(x-h)] = 0$$

for every $x \in \mathbb{R}^1$. Does this imply that f is continuous?

Solution. It does not imply that f is continuous. The following is a counterexample.

$$f(x) = \begin{cases} 1 & \text{if x=0} \\ 0 & \text{otherwise} \end{cases}.$$

Notice that for $\epsilon = \frac{1}{2}$, for any $\delta > 0$ and any x such that $0 < |0 - x| < \delta$ (i.e. $x \neq 0$), we have

$$|f(x) - f(0)| = |0 - 1| = 1 > \tfrac{1}{2},$$

so f is discontinuous at 0.

However, for any $x \neq 0$, note that for any $h < |x|$, $x + h \neq 0$ and $x - h \neq 0$, so

$$f(x+h) - f(x-h) = 0 - 0 = 0,$$

and thus $\lim_{h \to 0} [f(x+h) - f(x-h)] = 0$. If $x = 0$, then for any $h > 0$,

$$f(x+h) - f(x-h) = 0 - 0 = 0,$$

and thus $\lim_{h \to 0} [f(x+h) - f(x-h)] = 0$ again. This finishes the proof of the counterexample.

Problem 5 (WR Ch 4 #7)**.** If $E \subset X$ and f is a function defined on X, the *restriction* of f to E is the function g whose domain of definition is E, such that $g(p) = f(p)$ for $p \in E$. Define f and g on $\mathbb{R}^2$ by: $f(0,0) = g(0,0) = 0$, $f(x, y) = xy^2/(x^2 + y^4)$, $g(x, y) = xy^2/(x^2 + y^6)$ if $(x, y) \neq (0,0)$. Prove that f is bounded on $\mathbb{R}^2$, that g is unbounded in every neighborhood of $(0,0)$, and that f is not continuous at $(0,0)$; nevertheless, the restrictions of both f and g to every straight line in $\mathbb{R}^2$ are continuous!

Solution. f is bounded on $\mathbb{R}^2$ because

$$(|x| - y^2)^2 \geq 0 \iff x^2 - 2|x|y^2 + y^4 \geq 0 \iff x^2 + y^4 \geq 2|x|y^2 \iff \frac{1}{2} \geq \left|\frac{xy^2}{x^2 + y^4}\right|.$$

g is unbounded in every neighborhood of $(0,0)$ because if we let $x_n = \frac{1}{n^3}$ and $y_n = \frac{1}{n}$ then $(x_n, y_n) \to (0,0)$ and

$$g(x_n, y_n) = \frac{\frac{1}{n^3}\frac{1}{n^2}}{\frac{1}{n^6} + \frac{1}{n^6}} = \frac{\frac{1}{n^5}}{\frac{2}{n^6}} = \frac{n}{2} \to \infty.$$

This is because $(x_n, y_n) \to (0,0)$ implies that for any neighborhood U of $(0,0)$, there is some $N \in \mathbb{N}$ such that $(x_n, y_n) \in U$ for $n \geq N$, and $g(x_n, y_n)$ gets arbitrarily large. Next, we show f is not continuous at $p = (0,0)$ simply by using Theorems 4.2 and Theorem 4.6 to say

$$f \text{ is continuous at } p \overset{4.6}{\iff} \lim_{x \to p} f(x) = f(p) \overset{4.2}{\iff} f(p_n) \to f(p) \text{ for every sequence } p_n \to p.$$

So to show f is not continuous, we only need to find one sequence $p_n \to p$ such that $f(p_n) \not\to f(p)$. Let $x_n = \frac{1}{n^2}$ and $y_n = \frac{1}{n}$, and put $p_n = (x_n, y_n)$, so that $p_n \to p = (0,0)$. But then

$$f(p_n) = f(x_n, y_n) = \frac{\frac{1}{n^2}\frac{1}{n^2}}{\frac{1}{n^4} + \frac{1}{n^4}} = \frac{\frac{1}{n^4}}{\frac{2}{n^4}} = \tfrac{1}{2} \neq 0 = f(0,0) = f(p),$$

so f is not continuous at $(0,0)$.

Lastly, we show that the restriction of f and g to straight lines in $\mathbb{R}^2$ are continuous functions. First of all, f and g are continuous on $\mathbb{R}^2 \setminus \{(0,0)\}$ because they are composed of addition, multiplication, and division with a denominator that is nonzero on $\mathbb{R}^2 \setminus \{(0,0)\}$. Therefore f and g restricted to any line that does not go through $(0,0)$ are already continuous. What we need to check is that they are continuous at $(0,0)$ for any line going through $(0,0)$. There are two lines of this form: $y = mx$ and $x = 0$. Along the line $x = 0$, f and g are constantly 0, so they are continuous. Along the line $y = mx$, for $x \neq 0$, we have

$$f(x, mx) = \frac{x(mx)^2}{x^2 + (mx)^4} = \frac{m^2 x}{1 + m^4 x^2} \to 0 = f(0,0) \qquad \text{as } x \to 0,$$

and

$$g(x, mx) = \frac{x(mx)^2}{x^2 + (mx)^6} = \frac{m^2 x}{1 + m^6 x^2} \to 0 = g(0,0) \qquad \text{as } x \to 0,$$

so f and g are continuous at $(0,0)$ when restricted to straight lines.

Problem 6 (WR Ch 4 #8)**.** Let f be a real uniformly continuous function on the bounded set E in $\mathbb{R}^1$. Prove that f is bounded on E.

Show that the conclusion is false if boundedness of E is omitted from the hypothesis.

Solution. Since E is a bounded set in $\mathbb{R}$, there is some $R > 0$ such that $E \subset [-R, R]$. Set $\epsilon = 1$. Since f is uniformly continuous, there is some $\delta > 0$ such that $|x - y| < \delta$ implies $|f(x) - f(y)| < \epsilon = 1$. Let $n \in \mathbb{N}$ be the smallest integer such that $\frac{1}{n} < \delta$, and let

$$[-R, -R + \tfrac{1}{n}], [-R + \tfrac{1}{n}, -R + \tfrac{2}{n}], \ldots, [R - \tfrac{2}{n}, R - \tfrac{1}{n}], [R - \tfrac{1}{n}, R]$$

be a collection S of $2nR$ intervals that exactly line up and cover $[-R, R]$. By omitting some if necessary, let $I_1, I_2, \ldots, I_N$ be those intervals from the collection that overlap with E, i.e. such that $I_k \cap E \neq \emptyset$ for $1 \leq k \leq N$. Now for each k from 1 to N, pick some point $x_k \in I_k \cap E$. Then we have

$$x_k, x \in I_k \cap E \implies |x_k - x| < \tfrac{1}{n} < \delta \implies |f(x_k) - f(x)| < 1 \implies |f(x)| < 1 + |f(x_k)|.$$

Letting $M = \max_{1 \leq k \leq N} (1 + f(x_k))$, for any $x \in E$ we have $|f(x)| < M$, so f is a bounded function on E.

The second part asks us to show the conclusion is false if boundedness of E is omitted from the hypothesis. Let $E = \mathbb{R}$, which is unbounded, and let $f(x)$, which is uniformly continuous because for any $\epsilon > 0$, let $\delta = \epsilon$ and we have

$$|x - y| < \delta \implies |f(x) - f(y)| = |x - y| < \delta = \epsilon.$$

However, $f(x) = x$ is not bounded on $E = \mathbb{R}$.