Math 140a - HW 2 Solutions

Problem 1 (WR Ch 1 #2)**.** A complex number z is said to be *algebraic* if there are integers $a_0, \ldots, a_n$ not all zero, such that

$$a_0 z^n + a_1 z^{n-1} + \cdots + a_{n-1} z + a_n = 0. \tag{$*$}$$

Prove that the set of all algebraic numbers is countable.

Solution. Pick some $N \in \mathbb{N}$ and define

$$E_N = \left\{ z \in \mathbb{C} \;\middle|\; \begin{array}{l} \exists\, a_0, \ldots, a_n \in \mathbb{Z} \text{ not all zero such that} \\ (*) \text{ holds and } n + |a_0| + \cdots + |a_n| = N \end{array} \right\}.$$

Now if $n \geq N$ we have $N - n \leq 0$, but then

$$n + |a_0| + \cdots + |a_n| = N \qquad \Longrightarrow \qquad |a_0| + \cdots + |a_n| = N - n \leq 0,$$

a contradiction because the a_i's cannot be all zero. Therefore, we must have that $0 < n < N$. Next, for any $0 \leq i \leq n$ we also have

$$|a_i| \leq |a_0| + \cdots + |a_n| \leq n + |a_0| + \cdots + |a_n| = N.$$

Therefore, $|a_i| \leq N$ for $0 \leq i \leq n$. That means $a_i \in \{-N, -N+1, \ldots, -1, 0, 1, \ldots, N-1, N\}$ for $0 \leq i \leq n < N$, so that there are at most $2N+1$ choices for each a_i, and there are at most $N-1$ a_i's to chose for. Thus there are at most $(2N+1)^{N-1}$ choices for $a_0, \ldots, a_n$, and each choice will give a polynomial with at most n roots by the fundamental theorem of algebra (and remember n is at most $N-1$). Thus the number of elements of E_N is bounded by the number $(N-1)(2N+1)^{N-1}$, so E_N is a finite set for every $N \in \mathbb{N}$.

From here, we cite Theorem 2.12 to see that since the set of algebraic numbers can be written as $\bigcup_{N=1}^{\infty} E_N$, then the set of algebraic numbers is countable.

Problem 2 (WR Ch 1 #6)**.** Let E' be the set of all limit points of E. Prove that E' is closed. Prove that E and $\bar{E}$ have the same limit points. Do E and E' always have the same limit points?

Solution.
E' is closed if every limit point of E' is a point of E'. Let p be a limit point of E'. Then by definition we have that every neighborhood $N_r(p)$ of p contains a point $q \neq p$ such that $q \in E'$. Choose a real number $R > 0$ small enough that $R < d(p, q)$ and so that $N_R(q) \subset N_r(p)$. Since q is a limit point of E, there must be some $t \neq q$ in $N_R(q)$ such that $t \in E$. But then $t \in N_R(q) \subset N_r(p)$ and $t \neq p$ since $p \notin N_R(q)$. Therefore, we have shown that any neighborhood of p contains some $t \neq p$ with $t \in E$, so that p is a limit point of E.

Now we want to show that $E' = (\bar{E})'$. Let $p \in E'$. Then any neighborhood N of p contains some $q \neq p$ with $q \in E$. Since $E \subset \bar{E}$, then $q \in \bar{E}$, so $p \in (\bar{E})'$. This shows that $E' \subset (\bar{E})'$. To show the other direction of containment, let $p \in (\bar{E})'$. Then any neighborhood $N_r(p)$ contains some $q \neq p$ with $q \in \bar{E} = E \cup E'$. If $q \in E$, then $p \in E'$ and we are done. Otherwise, $q \in E'$. Choose a real number $R > 0$ small enough that $R < d(p,q)$ and so that $N_R(q) \subset N_r(p)$. Since q is a limit point of E, there must be some $t \neq q$ in $N_R(q)$ such that $t \in E$. But then $t \in N_R(q) \subset N_r(p)$ and $t \neq p$ since $p \notin N_R(q)$. Therefore, we have shown that any neighborhood of p contains some $t \neq p$ with $t \in E$, so that p is a limit point of E. Thus $E' \subset (\bar{E})'$, and we have shown that $E' = (\bar{E})'$

E and E' do not always have the same limit points, as proven by the existence of the following counterexample. Let $E = \{\frac{1}{n} | n \in \mathbb{N}\}$. Then $E' = \{0\}$ and $(E')' = \emptyset$.

Problem 3 (WR Ch 1 #9)**.** Let E° denote the set of all interior points of a set E.

(a) Prove that E° is always open.

Solution. An open set is one that contains all of its interior points. A point p is an interior point of E° if there exists some neighborhood N of p with $N \subset E^\circ$. But $E^\circ \subset E$, so that $N \subset E$. Hence $p \in E^\circ$. This proves that E° contains all of its interior points, and thus is open.

(b) Prove that E is open if and only if $E^\circ = E$.

Solution.

$\boxed{\Longrightarrow}$ If E is open, all of its points are interior points, so that $E \subset E^\circ$. Also, the set of interior points of E is a subset of the set of points of E, so that $E^\circ \subset E$. Thus $E^\circ = E$.

$\boxed{\Longleftarrow}$ If $E^\circ = E$, then every point of E is an interior point of E, so E is open.

(c) If $G \subset E$ and G is open, prove that $G \subset E^\circ$.

Solution. If p is an interior point of G, then there is some neighborhood N of p with $N \subset G$. Since $G \subset E$, $N \subset E$, which shows that p is an interior point of E. Thus $G^\circ \subset E^\circ$. If G is open, then by part (c) we know that $G = G^\circ$. Hence $G = G^\circ \subset E^\circ$.

(d) Prove that the complement of E° is the closure of the complement of E.

Solution.
$\boxed{(E^\circ)^c \subset \overline{E^c}}$ By taking compliments, we have that

$$(E^\circ)^c \subset \overline{E^c} \iff E^\circ \supset (\overline{(E^c)})^c.$$

Let $G = (\overline{(E^c)})^c$. G is the complement of the closed set $\overline{(E^c)}$, so it is open. Also, $E^c \subset \overline{(E^c)}$, which by taking complements of both sides implies that $E \supset (\overline{(E^c)})^c = G$. Therefore, by part (c), $G \subset E^\circ$, which we proved is equivalent to $(E^\circ)^c \subset \overline{E^c}$.

$\boxed{\overline{E^c} \subset (E^\circ)^c}$ Any closed set containing E^c must also contain $\overline{(E^c)}$ by Theorem 2.27(c). E° is open by part (a), and thus $(E^\circ)^c$ is closed by Theorem 2.23. Therefore, $\overline{E^c} \subset (E^\circ)^c$.

(e) Do E and $\bar{E}$ always have the same interiors?

Solution. No. Let $E = \mathbb{Q}$. Then $\bar{E} = \mathbb{R}$. The interior of E is $\emptyset$; the interior of $\bar{E}$ is $\mathbb{R}$.

(f) Do E and E° always have the same closures?

Solution. No. Let $E = \mathbb{Q}$. Then $E^\circ = \emptyset$. The closure of E is $\mathbb{R}$; the closure of E° is $\emptyset$.