MATH 140B - HW 8 SOLUTIONS

Problem 1 (WR Ch 8 #13)**.**
Put $f(x) = x$ if $0 \le x < 2\pi$, and apply Parseval's Theorem to conclude that

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

Solution. First we calculate the Fourier coefficients of $f(x) = x$ using integration by parts. For $n \neq 0$,

$$\begin{aligned}
c_n &= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-inx} \, dx \\
&= \frac{1}{2\pi} \int_{-\pi}^{\pi} x e^{-inx} \, dx \\
&= \frac{1}{2\pi} \left[x \frac{e^{-inx}}{-in} \right]_{-\pi}^{\pi} - \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{e^{-inx}}{-in} \, dx \\
&= \frac{1}{2\pi} \left[2\pi \frac{(-1)^n}{-in} \right] - \frac{1}{2\pi} \cancel{\left[\frac{e^{-inx}}{-n^2} \right]_{-\pi}^{\pi}} \\
&= i \frac{(-1)^n}{n}. \\
c_0 &= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) \, dx = \frac{1}{2\pi} \int_{-\pi}^{\pi} x \, dx = 0.
\end{aligned}$$

Parseval's identity says that

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} |f(x)|^2 \, dx = \sum_{-\infty}^{\infty} |c_n|^2.$$

Computing the right side we get

$$\sum_{-\infty}^{\infty} |c_n|^2 = \sum_{n \neq 0} \left| i \frac{(-1)^n}{n} \right|^2 = 2 \sum_{n=1}^{\infty} \frac{1}{n^2}.$$

And the left side is

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} |x|^2 \, dx = \frac{1}{2\pi} \left[\frac{x^3}{3} \right]_{-\pi}^{\pi} = \frac{1}{2\pi} \cdot \frac{2\pi^3}{3} = \frac{\pi^2}{3}.$$

Putting this together,

$$\frac{\pi^2}{3} = 2 \sum_{n=1}^{\infty} \frac{1}{n^2} \qquad \Longrightarrow \qquad \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

Problem 2 (WR Ch 8 #19).
Suppose f is a continuous function on $\mathbb{R}^1$, $f(x+2\pi) = f(x)$, and α/π is irrational. Prove that

$$\lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} f(x+n\alpha) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\,dt$$

for every x.

Solution. First we prove it for $f(x) = e^{ikx}$ with $k \in \mathbb{Z}$ and $k \neq 0$.

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{ikt}\,dt = \frac{1}{2\pi}\left[\frac{e^{ikx}}{ik}\right]_{-\pi}^{\pi} = 0.$$

Also,

$$\begin{aligned}\left|\frac{1}{N}\sum_{n=1}^{N} e^{ik(x+n\alpha)}\right| &= \frac{1}{N}\left|\sum_{n=1}^{N} e^{ikx}e^{ikn\alpha}\right| \\ &= \frac{1}{N}\left|e^{ikx}\sum_{n=1}^{N} e^{ikn\alpha}\right| \\ &= \frac{1}{N}\left|e^{ikx}\right|\left|\sum_{n=1}^{N} e^{ikn\alpha}\right| \\ &= \frac{1}{N}\left|\sum_{n=1}^{N} e^{ikn\alpha}\right| \\ &= \frac{1}{N}\left|\sum_{n=1}^{N} \left(e^{ik\alpha}\right)^n\right|,\end{aligned}$$

and since α/π is irrational, that means $k\alpha$ is not a multiple of π, so $e^{ik\alpha} \neq 1$, and we can use the partial sum formula from page 61 of Rudin to get

$$\left|\sum_{n=1}^{N}\left(e^{ik\alpha}\right)^n\right| = \left|\sum_{n=0}^{N}\left(e^{ik\alpha}\right)^n - 1\right| = \left|\frac{1-\left(e^{ik\alpha}\right)^{N+1}}{1-\left(e^{ik\alpha}\right)} - 1\right| < \frac{2}{|1-e^{ik\alpha}|} + 1 = M,$$

so the sum is bounded by M. Therefore,

$$\left|\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N} e^{ik(x+n\alpha)}\right| = \lim_{N\to\infty}\left|\frac{1}{N}\sum_{n=1}^{N} e^{ik(x+n\alpha)}\right| \leq \lim_{N\to\infty}\frac{M}{N} = 0.$$

This establishes the equality for $f(x) = e^{ikx}$ with $k \neq 0$. If $k = 0$, we have $f(x) = 1$, so

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N} f(x+n\alpha) = \lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N} 1 = \lim_{N\to\infty}\frac{1}{N}N = 1 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\,dt.$$

Now that we've shown it for all functions of the form e^{ikx}, we set some $\epsilon > 0$ and we use Theorem 8.15 to get a trigonometric polynomial $P(x) = \sum_{k=-K}^{K} c_k e^{ikx}$ such that $|P(x) - f(x)| < \epsilon/2$

for all real x. Then

$$\begin{aligned}\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N} f(x+n\alpha) &= \lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}\left(\left[f(x+n\alpha)-P(x+n\alpha)\right]+P(x+n\alpha)\right)\\ &= \lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}\left[f(x+n\alpha)-P(x+n\alpha)\right]+\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}P(x+n\alpha)\end{aligned}$$

$$\begin{aligned}\frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\,dt &= \frac{1}{2\pi}\int_{-\pi}^{\pi}[f(t)-P(t)]+P(t)\,dt\\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi}[f(t)-P(t)]+\frac{1}{2\pi}\int_{-\pi}^{\pi}P(t)\,dt.\end{aligned}$$

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}P(x+n\alpha)=\frac{1}{2\pi}\int_{-\pi}^{\pi}P(t)\,dt$$

$$\begin{aligned}&\left|\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N} f(x+n\alpha)\ -\frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\,dt\right|\\ &\le\left|\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}[f(x+n\alpha)-P(x+n\alpha)]-\frac{1}{2\pi}\int_{-\pi}^{\pi}[f(t)-P(t)]\,dt\right|\\ &\le\left|\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}[f(x+n\alpha)-P(x+n\alpha)]\right|+\left|\frac{1}{2\pi}\int_{-\pi}^{\pi}[f(t)-P(t)]\,dt\right|\\ &\le\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}|f(x+n\alpha)-P(x+n\alpha)|+\frac{1}{2\pi}\int_{-\pi}^{\pi}|f(t)-P(t)|\,dt\\ &<\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}\epsilon/2+\frac{1}{2\pi}\int_{-\pi}^{\pi}\epsilon/2\,dt\\ &=\frac{\epsilon}{2}+\frac{\epsilon}{2}=\epsilon.\end{aligned}$$

Since ϵ was arbitrary, this proves the equality.

Problem 3 (Supp-3 #1)**.** Suppose $f:[0,\pi]\to\mathbb{R}$ is continuous and $\int_0^\pi f(x)\sin nx=0$, $n=1,2,\ldots$. Does it follow that $f\equiv 0$? Proof or counterexample.

Solution. Let $c=\frac{1}{\pi}\int_0^\pi f(x)\,dx$ and define $g:[-\pi,\pi]\to\mathbb{R}$ by

$$g(x)=\begin{cases} f(x) & x\in[0,\pi],\\ -f(-x) & x\in[-\pi,0).\end{cases}$$

Then g is an odd function (i.e. $g(-x)=-g(x)$), and since $\cos nx$ is an even function, that means that $g(x)\cdot\cos nx$ is an odd function, and therefore

$$\int_{-\pi}^{\pi} g(x)\cos nx\,dx=0,$$

because integrating an odd function symmetrically around 0 gives 0. Also note that, (with a change of variables in the first integral using $u = -x$)

$$\begin{aligned}
\int_{-\pi}^{\pi} g(x)\sin nx\,dx &= \int_{-\pi}^{0} -f(-x)\sin nx\,dx + \int_{0}^{\pi} f(x)\sin nx\,dx \\
&= \int_{0}^{\pi} f(u)\sin nu\,du + \int_{0}^{\pi} f(x)\sin nx\,dx \\
&= 2\cancel{\int_{0}^{\pi} f(x)\sin nx\,dx} \\
&= 0.
\end{aligned}$$

Calculating the Fourier coefficients for g for $n \geq 0$:

$$\begin{aligned}
c_n &= \frac{1}{2\pi}\int_{-\pi}^{\pi} g(x)\,e^{-inx}\,dx \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} g(x)\,(\cos nx - i\sin nx)\,dx \\
&= \frac{1}{2\pi}\left(\cancel{\int_{-\pi}^{\pi} g(x)\cos nx\,dx} - i\cancel{\int_{-\pi}^{\pi} g(x)\sin nx\,dx}\right) \\
&= 0. \\
c_0 &= \frac{1}{2\pi}\int_{-\pi}^{\pi} g(x)\,dx \\
&= \frac{1}{2\pi}2\int_{0}^{\pi} (f(x) - c)\,dx \\
&= \frac{1}{\pi}\int_{0}^{\pi} f(x)\,dx - \frac{1}{\pi}\int_{0}^{\pi} c\,dx \\
&= \frac{1}{\pi}\int_{0}^{\pi} f(x)\,dx - c \\
&= 0.
\end{aligned}$$

Now by Parseval's identity,

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} |g(x)|^2\,dx = \sum_{-\infty}^{\infty} |c_n|^2 = 0,$$

and since g^2 is continuous and $g^2 \geq 0$, then we have $g^2 \equiv 0$ by Problem 6.2 proved on a previous homework. Therefore, $g \equiv 0$, so $f \equiv 0$.

Problem 4 (Supp-3 #3)**.** If f is real analytic in a neighborhood of x_0 and $f(x_0) = 0$, show that $f(x)/(x - x_0)$ is real analytic in the same neighborhood.

Solution. If f is real analytic in a neighborhood of x_0 and $f(x_0) = 0$ then there is some $\epsilon > 0$ such that for $x \in B(x_0;\epsilon)$,

$$f(x) = \sum_{n=0}^{\infty} c_n(x - x_0)^n \qquad \text{and} \qquad 0 = f(x_0) = c_0 \qquad \Longrightarrow \qquad f(x) = \sum_{n=1}^{\infty} c_n(x - x_0)^n.$$

Letting $f_k = \sum_{n=1}^{k} c_n(x-x_0)^n$, we have $f_k \rightrightarrows f$ on $B(x_0;\epsilon)$ because f is real analytic on $B(x_0;\epsilon)$. Also, by Theorem 8.1, $f_k' = \sum_{n=1}^{k} nc_n(x-x_0)^{n-1}$ is such that $\{f_k'\}$ converges uniformly to f' on $B(x_0;\epsilon)$ (this means the radius of convergence of f_k' is some number $R' \geq \epsilon$). We're going to prove that $f_k/(x-x_0)$ has the same radius of convergence as f_k', and that will finish the proof. Therefore,

$$\frac{f_k(x)}{x-x_0} = (x-x_0)^{-1}\sum_{n=1}^{k} c_n(x-x_0)^n = \sum_{n=1}^{k} c_n(x-x_0)^{n-1},$$

and checking the radius of convergence R of this last power series, we have

$$\frac{1}{R} = \limsup_{n\to\infty} \sqrt[n-1]{|c_n|} = \limsup_{n\to\infty} \sqrt[n-1]{n}\sqrt[n-1]{|c_n|} = \limsup_{n\to\infty} \sqrt[n-1]{n|c_n|} = \frac{1}{R'},$$

since $\lim_{n\to\infty} \sqrt[n-1]{n} = 1$. Then $R = R' \geq \epsilon$. That means $f_k/(x-x_0)$ converges uniformly on $B(x_0;\epsilon)$, and thus $f(x)/(x-x_0)$ is real analytic on $B(x_0;\epsilon)$.

Problem 5 (Supp #4)**.** Prove that if $f(x)$ is real analytic on (a,b) and $c \in (a,b)$, then $F(x) = \int_c^x f(t)\,dt$ is also real analytic on (a,b).

Solution. Let $x_0 = \frac{b+a}{2}$ (the point in the middle of a and b). If f is real analytic on (a,b) and $c \in (a,b)$, then letting $f_k(x) = \sum_{n=0}^{k} c_n(x-x_0)^n$ we have $f_k \rightrightarrows f$ on $[a,b]$. Let R_1 be the radius of convergence of f_k. What we have shown so far is that $R_1 \geq \left|\frac{b-a}{2}\right|$. Now we set

$$G_k(x) = \int_{x_0}^{x} f_k(t)\,dt = \int_{x_0}^{x}\sum_{n=0}^{k} c_n(t-x_0)^n\,dt = \sum_{n=0}^{k}\int_{x_0}^{x} c_n(t-x_0)^n\,dt = \sum_{n=0}^{k}\frac{c_n}{n+1}(x-x_0)^{n+1},$$

then the radius of convergence R_2 of this last power series is given by

$$\frac{1}{R_2} = \limsup_{n\to\infty}\sqrt[n+1]{\frac{|c_n|}{n+1}} = \limsup_{n\to\infty}\frac{1}{\sqrt[n+1]{n+1}}\sqrt[n+1]{|c_n|} = \limsup_{n\to\infty}\sqrt[n+1]{|c_n|} = \frac{1}{R_1},$$

since $\lim_{n\to\infty}\sqrt[n+1]{n+1} = 1$, and thus we have $R_2 = R_1 \geq \left|\frac{b-a}{2}\right|$, so G_k converges uniformly on $[a,b]$. If we let $G(x) = \int_{x_0}^{x} f(t)\,dt$, then by Theorem 7.16 we have $G_k \rightrightarrows G$ on $[a,b]$. Lastly, if we let $C = \int_c^{x_0} f(t)\,dt$, we have

$$F(x) = \int_c^x f(t)\,dt = \int_c^{x_0} f(t)\,dt + \int_{x_0}^{x} f(t)\,dt = G(x) + C,$$

and similarly we define $F_k(x) = G_k(x) + C$, so that $F_k \rightrightarrows F$, and thus F is real analytic on $[a,b]$.