MATH 140B - HW 7 SOLUTIONS

Problem 1 (WR Ch 8 #1)**.** Define

$$f(x) = \begin{cases} e^{-1/x^2} & (x \neq 0), \\ 0 & (x = 0). \end{cases}$$

Prove that f has derivatives of all orders at $x = 0$, and that $f^{(n)}(0) = 0$ for $n = 1, 2, 3, \ldots$.

Solution.

Claim 1. For any rational function $R(x)$, $\lim_{x\to 0} R(x)e^{-1/x^2} = 0$.

Let $R(x) = \frac{p(x)}{q(x)}$ for polynomials p and q. Let m be the smallest power of x in q. Then by dividing the top and bottom of the fraction $\frac{p(x)}{q(x)}$ by x^m, we have that the bottom does not vanish, and then the claim follows from the linearity of limits and Theorem 8.6(f), which says $\lim_{x\to\infty} x^n e^{-x} = 0$ for any $n \in \mathbb{Z}$.

Claim 2. For $x \neq 0$, $f^{(n)}(x) = R_n(x)e^{-1/x^2}$ for some rational function $R_n(x)$.

We prove this by induction. For $n = 1$, we have $f'(x) = e^{-1/x^2} \frac{d}{dx}\left(\frac{-1}{x^2}\right) = \frac{2}{x^3} e^{-1/x^2}$, which satisfies the requirement. Now assume for our induction hypothesis that $f^{(n)}(x) = \frac{p(x)}{q(x)} e^{-1/x^2}$. Then by the product rule and the quotient rule:

$$\begin{aligned} f^{(n+1)}(x) &= \frac{p(x)}{q(x)} \left(\frac{2}{x^3} e^{-1/x^2}\right) + \left(\frac{q(x)p'(x) - q'(x)p(x)}{q^2(x)}\right) e^{-1/x^2} \\ &= \left(\frac{2p(x)}{x^3 q(x)} + \frac{q(x)p'(x) - q'(x)p(x)}{q^2(x)}\right) e^{-1/x^2} \\ &= \left(\frac{2p(x)q(x) + x^3 q(x)p'(x) - x^3 q'(x)p(x)}{x^3 q^2(x)}\right) e^{-1/x^2} \end{aligned}$$

which satisfies the claim.

Now we solve the problem by induction using the two claims. For $n = 1$,

$$f'(0) = \lim_{h\to 0} \frac{f(h) - f(0)}{h} = \lim_{h\to 0} \frac{e^{-1/h^2} - 0}{h} = 0.$$

Assuming that $f^{(n)}(0) = 0$, we then have

$$f^{(n+1)}(0) = \lim_{h\to 0} \frac{f^{(n)}(h) - f^{(n)}(0)}{h} = \lim_{h\to 0} \frac{R_n(h)e^{-1/h^2} - 0}{h} = 0.$$

Problem 2 (WR Ch 8 #2)**.** Let a_{ij} be the number in the ith row and jth column of the array

$$\begin{array}{ccccc} -1 & 0 & 0 & 0 & \cdots \\ \frac{1}{2} & -1 & 0 & 0 & \cdots \\ \frac{1}{4} & \frac{1}{2} & -1 & 0 & \cdots \\ \frac{1}{8} & \frac{1}{4} & \frac{1}{2} & -1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{array}$$

so that

$$a_{ij} = \begin{cases} 0 & (i < j), \\ -1 & (i = j), \\ 2^{j-i} & (i > j). \end{cases}$$

Prove that

$$\sum_i \sum_j a_{ij} = -2, \qquad \sum_j \sum_i a_{ij} = 0.$$

Solution. Summing over columns first, we have

$$\text{for any } j, \quad \sum_i a_{ij} = -1 + \sum_{n=1}^{\infty} \tfrac{1}{2^n} = -1 + 1 = 0 \qquad \Longrightarrow \qquad \sum_j \sum_i a_{ij} = \sum_j 0 = 0.$$

Summing over rows first, using the identity $\sum_{n=0}^{N} x^n = \frac{1-x^{N+1}}{1-x}$, we have

$$\sum_j a_{ij} = -1 + \sum_{n=1}^{i-1} \frac{1}{2^n} - 1 = -1 + \frac{1 - \frac{1}{2^i}}{1 - \frac{1}{2}} = \frac{-1}{2^{i-1}} \qquad \Longrightarrow \qquad \sum_i \sum_j a_{ij} = \sum_i \frac{-1}{2^{i-1}} = -2.$$

Problem 3 (WR Ch 8 #5)**.**

(a) $$\lim_{x \to 0} \frac{e - (1+x)^{1/x}}{x}.$$

(b) $$\lim_{n \to \infty} \frac{n}{\log n} [n^{1/n} - 1].$$

Solution.

(a)

$$\begin{aligned}
\lim_{x\to 0}\frac{e-(1+x)^{1/x}}{x} &= \lim_{x\to 0}\frac{e-e^{\frac{1}{x}\log(1+x)}}{x}\\
&= \lim_{x\to 0}\frac{-e^{\frac{1}{x}\log(1+x)}\left(\frac{-1}{x^2}\log(1+x)+\frac{1}{x(x+1)}\right)}{1}\\
&= \lim_{x\to 0}-(1+x)^{1/x}\left(\frac{-\log(1+x)}{x^2}+\frac{1}{x(x+1)}\right)\\
&= \lim_{x\to 0}-(1+x)^{1/x}\ \lim_{x\to 0}\left(\frac{-(1+x)\log(1+x)+x}{x^2(x+1)}\right)\\
&= -e\ \lim_{x\to 0}\left(\frac{-\log(1+x)-\frac{1+x}{1+x}+1}{3x^2+2x}\right)\\
&= e\ \lim_{x\to 0}\left(\frac{\log(1+x)}{3x^2+2x}\right)\\
&= e\ \lim_{x\to 0}\left(\frac{\frac{1}{1+x}}{6x+2}\right)\\
&= \frac{e}{2}.
\end{aligned}$$

(b) Since $\frac{1}{n}\log n \to 0$ as $n\to\infty$, we have

$$\begin{aligned}
\lim_{n\to\infty}\frac{n}{\log n}[n^{1/n}-1] &= \lim_{n\to\infty}\frac{n}{\log n}\left[e^{\frac{1}{n}\log n}-1\right]\\
&= \lim_{n\to\infty}\frac{e^{\frac{1}{n}\log n}-1}{\frac{1}{n}\log n}\\
&= \lim_{h\to 0}\frac{e^h-1}{h}\\
&= \left(\frac{d}{dx}e^x\right)\Big|_{x=0}\\
&= e^0\\
&= 1.
\end{aligned}$$

Problem 4 (WR Ch 8 #6)**.** Suppose $f(x)f(y)=f(x+y)$ for all real x and y.

(a) Assuming that f is differentiable and not zero, prove that

$$f(x)=e^{cx}$$

where c is a constant.

(b) Prove the same thing, assuming only that f is continuous.

Solution. We simply prove part (b). Assuming f is not identically zero, there exists some x such that $f(x) \neq 0$. Then

$$f(0)f(x) = f(0+x) = f(x) \qquad \Longrightarrow \qquad f(0) = 1.$$

By continuity at 0 this means there is a neighborhood $(-\epsilon, \epsilon)$ (for $\epsilon > 0$) on which f is greater than 0. This means that for any $n, m \in \mathbb{Z}$ with $n > \frac{1}{\epsilon}$ we have

$$f(\tfrac{m}{n}) = \underbrace{f(\tfrac{1}{n}) \cdot \ldots \cdot f(\tfrac{1}{n})}_{m \text{ times}} = [f(\tfrac{1}{n})]^m > 0,$$

so f is positive on a dense set of $\mathbb{R}$ and thus is positive everywhere. Next, we let $c = \log f(1)$, and let $g(x) = f(x) - e^{cx}$, which is a continuous function. We will prove that g vanishes on a dense set, and thus by continuity must vanish everywhere. For any nonzero $n \in \mathbb{Z}$, by the uniqueness of the positive nth root, we have

$$f(\tfrac{1}{n})^n = f(1) = e^c = \left(e^{\frac{c}{n}}\right)^n \qquad \Longrightarrow \qquad f(\tfrac{1}{n}) = e^{\frac{c}{n}} \qquad \Longrightarrow \qquad g(\tfrac{1}{n}) = 0.$$

Now for any $m \in \mathbb{N}$ and nonzero $n \in \mathbb{Z}$

$$g(\tfrac{m}{n}) = f(\tfrac{m}{n}) - e^{c\frac{m}{n}} = \left(f(\tfrac{1}{n})\right)^m - \left(e^{\frac{c}{n}}\right)^m = 0.$$