MATH 140B - HW 6 SOLUTIONS

Problem 1 (WR Ch 7 #18)**.** Let $\{f_n\}$ be a uniformly bounded sequence of functions which are Riemann-integrable on $[a,b]$, and put

$$F_n(x) = \int_a^x f_n(t)\,dt \qquad (a \le x \le b).$$

Prove that there exists a subsequence $\{F_{n_k}\}$ which converges uniformly on $[a,b]$.

Solution. By Theorem 7.25, all we need to show is that $\{F_n\}$ is pointwise bounded and equicontinuous. Since $\{f_n\}$ is uniformly bounded, there exists some $M > 0$ such that $|f_n(t)| < M$ for all $t \in [a,b]$ and all n. Therefore,

$$|F_n(x)| = \left|\int_a^x f_n(t)\,dt\right| \le \int_a^x |f_n(t)|\,dt \le \int_a^x M\,dt = M(x-a) \qquad \text{for all } n,$$

which proves pointwise boundedness of $\{F_n\}$. To prove equicontinuity, given some $\epsilon > 0$, we choose $\delta = \epsilon/M$ so that for any $x, y \in [a,b]$ such that $|x-y| < \delta$, we have

$$|F_n(x) - F_n(y)| = \left|\int_a^x f_n(t)\,dt - \int_a^y f_n(t)\,dt\right| = \left|\int_y^x f_n(t)\,dt\right| \le \int_y^x |f_n(t)|\,dt < M\delta = \epsilon.$$

Problem 2 (WR Ch 7 #19)**.** Let K be a compact metric space, let S be a subset of $\mathscr{C}(K)$. Prove that S is compact (with respect to the metric defined in Section 7.14) if and only if S is uniformly closed, pointwise bounded, and equicontinuous. (If S is not equicontinuous, then S contains a sequence which has no equicontinuous subsequence, hence has no subsequence that converges uniformly on K.)

Solution.

$\boxed{\Longrightarrow}$ Assume S is compact in $\mathscr{C}(K)$. By Theorem 2.34, S is uniformly closed. Let $x \in K$. Define the sets

$$U_n = \{f \in \mathscr{C}(K) : \|f\| < n\},$$

(which are the open balls of radius n around the function $g \equiv 0$ in the uniform metric). Notice that $\bigcup U_n \supset \mathscr{C}(K) \supset S$. By compactness (and the fact that $U_n \subset U_{n+1}$), there is some N such that $S \subset U_N$, which means $\|f\| \le N$ for all $f \in S$. This means that S is uniformly bounded, and thus pointwise bounded.

Lastly, we need to check equicontinuity. Set $\epsilon > 0$. First we want to prove that there are some finite number of functions $f_1, \ldots, f_n \in S$ such that for each $f \in S$, $\|f - f_i\| < \epsilon/3$ for some i. Assume otherwise. Then the balls $B(f; \epsilon/3)$ form an open cover of S which does not have a finite subcover, contradicting compactness.

Next, since each f_i for $1 \le i \le n$ is continuous on a compact set K, each one is uniformly continuous, so that there exists some $\delta_i > 0$ such that $|f_i(x) - f_i(y)| < \frac{\epsilon}{3}$ whenever $d(x, y) < \delta_i$.

Now, given any $f \in S$, we choose i such that $\|f - f_i\| < \frac{\epsilon}{3}$, and then for $\delta = \max\{\delta_1, \ldots, \delta_n\}$ we have

$$|x-y| < \delta \quad \Longrightarrow \quad |f(x)-f(y)| \le |f(x)-f_i(x)|+|f_i(x)-f_i(y)|+|f_i(y)-f(y)| < \frac{\epsilon}{3}+\frac{\epsilon}{3}+\frac{\epsilon}{3} = \epsilon.$$

$\boxed{\Longleftarrow}$ Assume S is uniformly closed, pointwise bounded, and equicontinuous. By Theorem 7.25, any infinite subset of S has a limit point, and since S is uniformly closed, this limit point is in S. Therefore, by Exercise 2.26 (which we proved last quarter), S is compact.

Problem 3 (WR Ch 7 #20). If f is continuous on $[0, 1]$ and if

$$\int_0^1 f(x)\, x^n \, dx = 0 \qquad (n = 0, 1, 2, \ldots),$$

prove that $f(x) = 0$ on $[0, 1]$.

Solution. Since f is continuous on a compact set, it is bounded. So there exists some M such that $|f(x)| \le M$ on $[0, 1]$. By the Weierstrass Approximation Theorem, there exists some sequence of polynomials $\{p_n\}$ such that $p_n \rightrightarrows f$. Thus or any $\epsilon > 0$ we can choose some $N \in \mathbb{N}$ s.t. $|f(x) - p_n(x)| < \frac{\epsilon}{M}$ for $n \ge N$. But this means that

$$|f(x)p_n(x) - f^2(x)| = |f(x)|\,|f(x) - p_n(x)| \le M|f(x) - p_n(x)| < \epsilon \qquad \text{for } n \ge N,$$

so $f p_n \rightrightarrows f^2$. At this point we should mention that, by the linearity of the integral, $\int_0^1 f(x)p(x)\,dx = 0$ for any polynomial. Therefore by Theorem 7.16 we have

$$\int_0^1 f^2(x)\,dx = \lim_{n\to\infty} \int_0^1 f(x)p_n(x)\,dx = \lim_{n\to\infty} 0 = 0,$$

and thus $f(x) = 0$ on $[0, 1]$ by Exercise 6.2.

Problem 4 (WR Ch 7 #22). Assume $f \in \mathscr{R}(\alpha)$ on $[a,b]$, and prove that there are polynomials P_n such that

$$\lim_{n\to\infty}\int_a^b |f-P_n|^2\,d\alpha = 0.$$

Solution. We need to show that for every $\epsilon > 0$ there exists some polynomial $P(x)$ such that $\int_a^b |f-P|^2\,d\alpha < \epsilon$. Since $f \in \mathscr{R}(\alpha)$ on $[a,b]$, by Exercise 6.12 there exists a continuous function g such that

$$\left\{\int_a^b |f-g|^2\,d\alpha\right\}^{1/2} < \sqrt{\frac{\epsilon}{2}}.$$

By the Weierstrass Approximation Theorem there exists a polynomial $P(X)$ such that

$$\|g-p\| < \sqrt{\frac{\epsilon}{2\int_a^b d\alpha}}.$$

Putting this together, we have

$$\int_a^b |f-P|^2\,d\alpha \le \int_a^b |f-g|^2\,d\alpha + \int_a^b |g-P|^2\,d\alpha < \frac{\epsilon}{2} + \frac{\epsilon}{2\int_a^b d\alpha}\int_a^b d\alpha = \epsilon.$$