MATH 140B - HW 3 SOLUTIONS

Problem 1. Suppose f is a real, continuous differentiable function on $[a,b]$ with $f(a) = f(b) = 0$, and

$$\int_a^b f^2(x)\,dx = 1.$$

Show that

$$\int_a^b x\,f(x)\,f'(x)\,dx = -1/2.$$

Solution. By integration by parts with

$$u = x\,f(x) \qquad dv = f'(x)\,dx$$
$$du = [x\,f'(x) + f(x)]\,dx \qquad v = f(x),$$

we have

$$\begin{aligned}
\int_a^b x\,f(x)\,f'(x)\,dx &= \cancel{\left[x f^2(x)\right]_a^b} - \int_a^b f(x)[x\,f'(x) + f(x)]\,dx \\
&= -\int_a^b x\,f(x)\,f'(x)\,dx - \int_a^b f^2(x)\,dx \\
&= -\int_a^b x\,f(x)\,f'(x)\,dx - 1 \\
2\int_a^b x\,f(x)\,f'(x)\,dx &= -1 \\
\int_a^b x\,f(x)\,f'(x)\,dx &= -1/2.
\end{aligned}$$

Problem 2. Prove that if $f \in \mathscr{R}[a,b]$ and g is a function for which $g(x) = f(x)$ for all x except for a finite number of points, then g is Riemann integrable. Is the result still true if $g(x) = f(x)$ for all x except for a countable number of points?

Solution. Set $\epsilon > 0$. Since $f \in \mathscr{R}[a,b]$, by Theorem 6.6 there exists a partition $P = \{a = x_0, x_1, \ldots, x_{n-1}, x_n = b\}$ such that

$$U(P,f) - L(P,f) < \frac{\epsilon}{3}.$$

Let the finite set on which g differs from f be $C = \{c_1, \ldots, c_m\}$ and let $M = \max_{c \in C} |g(c) - f(c)|$. We define the refinement P_k of P inductively by letting $P_0 = P$ and once we have defined P_{k-1}, we define P_k by adding the points $(c_i - \frac{1}{n})$ and $(c_i + \frac{1}{n})$ for each $1 \le i \le m$ if they are in $[a,b]$. Let n_k be the number of points in P_k. Then we have

$$\begin{aligned} U(P_k, g) &= \sum_{j=1}^{n_k - 1} \left(\sup_{x_{j-1} \le x \le x_j} g(x) \right) \Delta x_j \\ &\le \sum_{j=1}^{n_k - 1} \left(\sup_{x_{j-1} \le x \le x_j} f(x) \right) \Delta x_j + \sum_{\substack{\exists\ c \in C \text{ s.t.} \\ c \in (x_{k-1}, x_k)}} |g(c) - f(c)| \Delta x_j \\ &\le \sum_{j=1}^{n_k - 1} \left(\sup_{x_{j-1} \le x \le x_j} f(x) \right) \Delta x_j + \frac{mM}{k} \\ &= U(P_k, f) + \frac{mM}{k}. \end{aligned}$$

Choose the K to be the smallest integer larger than $\frac{3mM}{\epsilon}$. That way $\frac{mM}{k} < \frac{\epsilon}{3}$ for $k \ge K$. Then for the lower Riemann sum we have:

$$\begin{aligned} L(P_k, g) &= \sum_{j=1}^{n_k - 1} \left(\inf_{x_{j-1} \le x \le x_j} g(x) \right) \Delta x_j \\ &\ge \sum_{j=1}^{n_k - 1} \left(\inf_{x_{j-1} \le x \le x_j} f(x) \right) \Delta x_j - \sum_{\substack{\exists\ c \in C \text{ s.t.} \\ c \in (x_{k-1}, x_k)}} |g(c) - f(c)| \Delta x_j \\ &\ge \sum_{j=1}^{n_k - 1} \left(\inf_{x_{j-1} \le x \le x_j} f(x) \right) \Delta x_j - \frac{mM}{k} \\ &= L(P_k, f) - \frac{mM}{k}. \end{aligned}$$

Because P_K is a refinement of P, we have

$$U(P_K, f) - L(P_K, f) \le U(P, f) - L(P, f) < \epsilon,$$

and finally putting this all together,

$$\begin{aligned} U(P_K, g) - L(P_K, g) &\le \big(U(P_K, g) - U(P_K, f)\big) + \big(U(P_K, f) - L(P_K, f)\big) + \big(L(P_K, f) - L(P_K, g)\big) \\ &< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon. \end{aligned}$$

Therefore $g \in \mathscr{R}[a,b]$ by Theorem 6.6.

The result is not true if $g(x) = f(x)$ for all x except for a countable number of points by the following counterexample. Let

$$g(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \notin \mathbb{Q} \end{cases}.$$

Then $g \notin \mathscr{R}[a,b]$ but $g(x) = 0$ for except for a countable number of points (the rationals in $[a,b]$) and $0 \in \mathscr{R}[a,b]$.

Problem 3. Let $f : [0,\infty) \to \mathbb{R}$ be defined as $f(x) = 0$ if $0 \le x \le 1/2$ and $f(x) = 1$ if $1/2 \le x < \infty$. Show that the function

$$F(x) = \int_0^x f(t)\,dt,$$

defined for $0 \le x < \infty$, is differentiable for $x \ne 1/2$ and is not differentiable for $x = 1/2$.

Solution. f is continuous except at $x = 1/2$. By Theorem 6.20, this means that F is differentiable everywhere except possibly at $x = 1/2$, and $F'(x) = f(x)$ when $x \ne 1/2$.

$$\begin{aligned}
\lim_{h\to 0+} \frac{F(1/2+h) - F(1/2)}{h} &= \lim_{h\to 0+} \frac{\int_0^{1/2+h} f(t)\,dt - \int_0^{1/2} f(t)\,dt}{h} \\
&= \lim_{h\to 0+} \frac{1}{h} \int_{1/2}^{1/2+h} f(t)\,dt \\
&= \lim_{h\to 0+} \frac{1}{h}\; h \\
&= 1. \\
\lim_{h\to 0-} \frac{F(1/2+h) - F(1/2)}{h} &= \lim_{h\to 0-} \frac{\int_0^{1/2+h} f(t)\,dt - \int_0^{1/2} f(t)\,dt}{h} \\
&= \lim_{h\to 0-} \frac{1}{h} \int_{1/2}^{1/2+h} f(t)\,dt \\
&= \lim_{h\to 0-} \frac{1}{h}\; 0 \\
&= 0.
\end{aligned}$$

Therefore the limit does not exist, so F is not differentiable at $x = 1/2$.

Problem 4. Prove that if f and g are Riemann integrable on $[a,b]$ (i.e. $f, g \in \mathscr{R}[a,b]$) and there exists $N > 0$ such that $g(x) \ge 1/N$ for all $x \in [a,b]$, then $f/g \in \mathscr{R}[a,b]$.

Solution. Let g be bounded above by M on $[a,b]$. By Theorem 6.11, since $\phi(x) = \frac{1}{x}$ is continuous on $[\frac{1}{N}, M]$ (because it doesn't contain 0), then $\phi(g(x)) = \frac{1}{g(x)}$ is Riemann integrable on $[a,b]$. Now by Theorem 6.13(a), $f/g = f \cdot \frac{1}{g}$ is Riemann integrable on $[a,b]$.