MATH 140B - HW 2 SOLUTIONS

Problem 1 (WR Ch 5 #11)**.** Suppose f is defined in a neighborhood of x, and suppose $f''(x)$ exists. Show that

$$\lim_{h\to 0} \frac{f(x+h)+f(x-h)-2f(x)}{h^2} = f''(x).$$

Show by an example that the limit may exist even if $f''(x)$ does not.

Solution. Since $f'(x) = \lim_{h\to 0} \frac{f'(x+h)-f'(x)}{h}$, and since $h \to 0$ iff $-h \to 0$, if we replace h by $-h$ in this expression, we have

$$f'(x) = \lim_{(-h)\to 0} \frac{f'(x+(-h))-f'(x)}{(-h)} = \lim_{h\to 0} \frac{f'(x)-f'(x-h)}{h}.$$

Therefore,

$$f''(x) = \lim_{h\to 0} \frac{f'(x+h)-f'(x)}{h} = \lim_{h\to 0} \frac{\lim\limits_{h_1\to 0} \frac{f(x+h)-f(x+h-h_1)}{h_1} - \lim\limits_{h_2\to 0} \frac{f(x)-f(x-h_2)}{h_2}}{h},$$

and letting $h_1 = h = h_2$, i.e. taking them all to zero at the same rate (which we can do by Theorem 4.2), we have

$$f''(x) = \lim_{h\to 0} \frac{\frac{f(x+h)-f(x+h-h)}{h} - \frac{f(x)-f(x-h)}{h}}{h} = \lim_{h\to 0} \frac{f(x+h)+f(x-h)-2f(x)}{h^2}.$$

For the second portion, we would like to find an f so that $f'(x) = |x|$. One such choice of f could be $f(x) = \int_0^x |t|\,dt = \frac{1}{2}x|x|$. Now let $x = 0$. By continuity of f at $x = 0$ we have

$$\lim_{h\to 0} \left[f(x+h)+f(x-h)-2f(x)\right] = f(0)+f(0)-2f(0) = 0.$$

Also, clearly $\lim_{h\to 0} h^2 = 0$. So by L'Hôpital's rule we have

$$\lim_{h\to 0} \frac{f(x+h)+f(x-h)-2f(x)}{h^2} \overset{\text{L'H}}{=} \lim_{h\to 0} \frac{f'(x+h)-f'(x-h)}{2h} = \lim_{h\to 0} \frac{|0+h|-|0-h|}{2h} = 0.$$

That means the limit exists at $x = 0$, but $f'(x) = |x|$ is not differentiable at 0, so $f''(0)$ does not exist.

Problem 2 (WR Ch 5 #12)**.** If $f(x) = |x|^3$, compute $f'(x)$, $f''(x)$ for all real x, and show that $f^{(3)}(0)$ does not exist.

Solution. For $x \neq 0$, $|x|$ is a differentiable function with derivative

$$\operatorname{sgn}(x) = \begin{cases} 1 & \text{if } x > 0 \\ -1 & \text{if } x < 0 \end{cases}.$$

Thus by the chain rule in the first line and by the product rule in the second line,

$$f'(x) = 3|x|^2 \operatorname{sgn}(x) = 3x|x|.$$
$$f''(x) = 3|x| + 3x \operatorname{sgn}(x) = 3|x| + 3|x| = 6|x|.$$

Checking the cases for $x = 0$ by hand, we have

$$f'(0) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{|h|^3 - 0}{h} = \lim_{h\to 0} h|h| = 0.$$
$$f''(0) = \lim_{h\to 0} \frac{f'(x+h) - f'(x)}{h} = \lim_{h\to 0} \frac{3h|h| - 0}{h} = \lim_{h\to 0} 3|h| = 0.$$
$$f'''(0) = \lim_{h\to 0} \frac{f''(x+h) - f''(x)}{h} = \lim_{h\to 0} \frac{6|h| - 0}{h} = 6 \lim_{h\to 0} \frac{|h|}{h} = \text{DNE}$$

Problem 3 (WR Ch 6 #2)**.** Suppose $f \geq 0$, f is continuous on $[a, b]$, and $\int_a^b f(x)\,dx = 0$. Prove that $f(x) = 0$ for all $x \in [a, b]$.

Solution. Assume by way of contradiction that there is some $y \in [a, b]$ such that $f(y) > 0$, and let $\epsilon = \frac{f(y)}{2}$. Since f is continuous, there exists a $\delta > 0$ such that

$$0 < |y - x| < \delta \qquad \Longrightarrow \qquad |f(y) - f(x)| < \epsilon = \frac{f(y)}{2},$$

for $x \in [a, b]$. This last inequality gives us

$$f(y) - f(x) \leq |f(y) - f(x)| < \frac{f(y)}{2} \qquad \Longrightarrow \qquad f(x) > \frac{f(y)}{2} > 0.$$

Let $I = (y - \delta, y + \delta) \cap [a, b]$. What we have shown so far is that if $x \in I$, then $f(x) > \frac{f(y)}{2} > 0$. Now, given some partition P of $[a, b]$, we make a refinement P^* by adding if necessary (and if possible) a point in $(y - \delta, y) \cap [a, b]$ and a point in $(y, y + \delta) \cap [a, b]$ so that $y \in (x_{k-1}, x_k) \subset I$ with $x_{k-1}, x_k \in P^*$ for some $1 \leq k \leq n$. Then

$$0 = \underline{\int_a^b} f(x)\,dx = \sup_P L(P, f) \geq L(P^*, f) = \sum_{i=1}^{n} \left(\inf_{x_{i-1} \leq x \leq x_i} f(x) \right) \Delta x_i \geq \frac{f(y)}{2} \Delta x_k > 0,$$

a contradiction.

Problem 4 (WR Ch 6 #4). If $f(x) = 0$ for all irrational x, $f(x) = 1$ for all rational x, prove that $f \notin \mathscr{R}$ on $[a,b]$ for any $a < b$.

Solution. For any partition $P = \{a = x_0, x_1, \ldots, x_{n-1}, x_n = b\}$, we have

$$L(P,f) = \sum_{i=1}^{n} \left(\inf_{x_{i-1} \le x \le x_i} f(x) \right) \Delta x_i = \sum_{i=1}^{n} (0) \Delta x_i = 0 \qquad \text{by the density of } \mathbb{Q}^c,$$

$$U(P,f) = \sum_{i=1}^{n} \left(\sup_{x_{i-1} \le x \le x_i} f(x) \right) \Delta x_i = \sum_{i=1}^{n} (1) \Delta x_i = (b-a) \qquad \text{by the density of } \mathbb{Q}.$$

$$\underline{\int_a^b} f = \sup_P L(P,f) = 0 \neq (b-a) = \inf_P U(P,f) = \overline{\int_a^b} f,$$

so $f \notin \mathscr{R}$ on $[a,b]$.

Problem 5 (WR Ch 6 #5). Suppose f is a bounded real function on $[a,b]$, and $f^2 \in \mathscr{R}$ on $[a,b]$. Does it follow that $f \in \mathscr{R}$? Does the answer change if we assume that $f^3 \in \mathscr{R}$?

Solution. In the first case, we have the following counterexample. Let

$$f(x) = \begin{cases} 1 & \text{if } x \in (\mathbb{Q}^c \cap [a,b]) \\ -1 & \text{if } x \in (\mathbb{Q} \cap [a,b]) \end{cases}.$$

Then by the previous proof with -1 in place of 0, $f \notin \mathscr{R}$ on $[a,b]$. But $f^2 \equiv 1 \in \mathscr{R}$ on $[a,b]$. So it does **not** necessarily follow that if $f^2 \in \mathscr{R}$ on $[a,b]$ then $f \in \mathscr{R}$.

In the second case, it **does** necessarily follow that if $f^3 \in \mathscr{R}$ on $[a,b]$ then $f \in \mathscr{R}$ by the following proof. The reason this works for f^3 and not for f^2 is that the inverse of the cube function on $\mathbb{R}$ is well-defined and is $\phi(x) = \sqrt[3]{x}$. The square function does not have a well-defined inverse on all of $\mathbb{R}$ (since if $y = x^2$ then $x = \pm\sqrt{y}$).

By Theorem 6.11, since ϕ is continuous on all of $\mathbb{R}$, then

$$\phi(f^3(x)) = \sqrt[3]{f^3(x)} = f(x) \quad \text{is in } \mathscr{R} \text{ on } [a,b].$$

Problem 6 (WR Ch 6 #10). Let p and q be positive real numbers such that

$$\frac{1}{p} + \frac{1}{q} = 1.$$

Prove the following statements.

(a) If $u \geq 0$ and $v \geq 0$, then

$$uv \leq \frac{u^p}{p} + \frac{v^q}{q}.$$

Equality holds if and only if $u^p = v^q$.

(b) If $f \in \mathscr{R}(\alpha)$, $g \in \mathscr{R}(\alpha)$, $f \geq 0$, $g \geq 0$, and

$$\int_a^b f^p \, d\alpha = 1 = \int_a^b g^q \, d\alpha,$$

then

$$\int_a^b fg \, d\alpha = 1.$$

(c) If f and g are complex functions in $\mathscr{R}(\alpha)$, then

$$\left| \int_a^b fg \, d\alpha \right| \leq \left\{ \int_a^b |f|^p \, d\alpha \right\}^{1/p} \left\{ \int_a^b |g|^q \, d\alpha \right\}^{1/q}.$$

This is *Hölder's inequality*. When $p = q = 2$ it is usually called the Schwarz inequality.

(d) Show that Hölder's inequality is also true for the "improper" integrals described in Exercises 6.7 and 6.8.

Solution.

Claim. $f(x) = e^x$ is a convex function.

Let $x < t < y$. By the Mean Value Theorem, there exists some $a \in (x, t)$ such that

$$f(t) - f(x) = (t - x) f'(a) \qquad \text{which means} \qquad f'(a) = \frac{f(t) - f(x)}{t - x}.$$

Once again, by the Mean Value Theorem, there exists some $b \in (t, y)$ such that

$$f(y) - f(t) = (y - t) f'(b) \qquad \text{which means} \qquad f'(b) = \frac{f(y) - f(t)}{y - t}.$$

Notice that $f''(x) = e^x > 0$ for all $x \in \mathbb{R}$. This means that $f'(x)$ is strictly increasing. Therefore, since $a < b$, we have $f'(a) \leq f'(b)$, so

$$\frac{f(t) - f(x)}{t - x} = f'(a) \leq f'(b) = \frac{f(y) - f(t)}{y - t}.$$

Now for any $\lambda \in (0, 1)$ we have $x < (\lambda x + (1 - \lambda) y) < y$, so letting $t = (\lambda x + (1 - \lambda) y)$ the above inequality becomes

$$\begin{aligned}
\frac{f(t)-f(x)}{(\lambda x+(1-\lambda)y)-x} &\le \frac{f(y)-f(t)}{y-(\lambda x+(1-\lambda)y)} \\
\frac{f(t)-f(x)}{(1-\lambda)\cancel{(y-x)}} &\le \frac{f(y)-f(t)}{\lambda\cancel{(y-x)}} \\
\lambda f(t)-\lambda f(x) &\le (1-\lambda)f(y)-(1-\lambda)f(t) \\
f(t) &\le \lambda f(x)+(1-\lambda)f(y) \\
f(\lambda x+(1-\lambda)y) &\le \lambda f(x)+(1-\lambda)f(y),
\end{aligned}$$

so $f(x)=e^x$ is convex.

(a) From here, we let $\lambda=\frac{1}{p}$, so that $(1-\lambda)=\frac{1}{q}$. The desired result is trivial if $u=0$ or $v=0$, so assume they are both strictly positive. Letting $x=\log u^p$ and $y=\log v^q$, the above inequality becomes

$$\begin{aligned}
e^{\frac{1}{p}\log u^p+\frac{1}{q}\log v^q} &\le \tfrac{1}{p}e^{\log u^p}+\tfrac{1}{q}e^{\log v^q} \\
e^{\log u+\log v} &\le \tfrac{u^p}{p}+\tfrac{v^q}{q} \\
uv &\le \tfrac{u^p}{p}+\tfrac{v^q}{q}.
\end{aligned}$$

(b) By part (a), for every $x\in[a,b]$ we have

$$f(x)\,g(x)\le \frac{(f(x))^p}{p}+\frac{(g(x))^q}{q}.$$

Therefore, taking integrals, we have

$$\int_a^b fg\,d\alpha \le \frac{\int_a^b f^p\,d\alpha}{p}+\frac{\int_a^b g^q\,d\alpha}{q}=\frac{1}{p}+\frac{1}{q}=1.$$

(c) If $\int_a^b |f|\,d\alpha=0$ or $\int_a^b |g|\,d\alpha=0$ the inequality is trivial. Otherwise, let $A=\left\{\int_a^b |f|^p\,d\alpha\right\}^{1/p}>0$ and let $B=\left\{\int_a^b |g|^q\,d\alpha\right\}^{1/q}>0$, and let

$$F(x)=\frac{|f(x)|}{A} \qquad \text{and} \qquad G(x)=\frac{|g(x)|}{B}.$$

These functions satisfy the hypotheses of part (b), so

$$\begin{aligned}
\int_a^b FG\,d\alpha &\le 1 \\
\int_a^b \frac{|f|}{A}\frac{|g|}{B}\,d\alpha &\le 1 \\
\left|\int_a^b fg\,d\alpha\right| \le \int_a^b |f||g|\,d\alpha \le AB &= \left\{\int_a^b |f|^p\,d\alpha\right\}^{1/p}\left\{\int_a^b |g|^q\,d\alpha\right\}^{1/q}.
\end{aligned}$$

(c) Since $x \mapsto |x|$ and $x \mapsto x^{1/p}$ and $x \mapsto x^{1/q}$ are continuous functions (for $x > 0$), we have

$$\begin{aligned}
\left|\int_0^1 fg\,d\alpha\right| &= \left|\lim_{c\to 0}\int_c^1 fg\,d\alpha\right| = \lim_{c\to 0}\left|\int_c^1 fg\,d\alpha\right| \\
&\le \lim_{c\to 0}\left(\left\{\int_c^1 |f|^p\,d\alpha\right\}^{1/p}\left\{\int_c^1 |g|^q\,d\alpha\right\}^{1/q}\right) \\
&= \left(\lim_{c\to 0}\left\{\int_c^1 |f|^p\,d\alpha\right\}^{1/p}\right)\cdot\left(\lim_{c\to 0}\left\{\int_c^1 |g|^q\,d\alpha\right\}^{1/q}\right) \\
&= \left\{\lim_{c\to 0}\int_c^1 |f|^p\,d\alpha\right\}^{1/p}\cdot\left\{\lim_{c\to 0}\int_c^1 |g|^q\,d\alpha\right\}^{1/q} \\
&= \left\{\int_0^1 |f|^p\,d\alpha\right\}^{1/p}\left\{\int_0^1 |g|^q\,d\alpha\right\}^{1/q},
\end{aligned}$$

assuming the integrals are all nonzero and finite. If they are not, the inequality is trivial. The proof follows similarly for $\int_a^\infty$.