A Perron-type Theorem on the Principal Eigenvalue of Nonsymmetric Elliptic Operators

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And I cherish more than anything else the Analogies,
my most trustworthy masters.
They know all the secrets of Nature. —Kepler

Abstract. We provide a proof for a Perron-type theorem on the principal eigenvalue of nonsymmetric elliptic operators based on the strong maximum principle. This proof is modeled after a variational proof of Perron's theorem for matrices with positive entries that does not appeal to Perron–Frobenius theory.

1. INTRODUCTION. Perron's theorem (cf. [**3**, Theorem 1, Ch. 13]) asserts that a square matrix $A = (\alpha_{ij})$ with positive entries $\alpha_{ij} > 0$ must possess a positive eigenvalue with multiplicity one. Moreover, for this positive eigenvalue, there exists an eigenvector whose entries are all positive. The purpose of this note is to prove an analogous result for second order elliptic operators, which we will now describe.

Let Ω be a smooth bounded domain in $\mathbb{R}^n$ and let

$$L = -\sum_{i,j=1}^{n} a^{ij}(x)\frac{\partial^2}{\partial x_i \partial x_j} + \sum_{k=1}^{n} b^k(x)\frac{\partial}{\partial x_k} + c(x)$$

be an elliptic operator defined on Ω. For simplicity, we assume that $a^{ij}(x), b^k(x), c(x) \in C^\infty(\overline{\Omega})$. We also assume that L is *uniformly elliptic*, i.e., there exists a constant $\theta > 0$ so that for all $x \in \overline{\Omega}$,

$$\sum_{i,j=1}^{n} a^{ij}(x)\xi_i\xi_j \geq \theta|\xi|^2 \tag{1}$$

for any $\xi \in \mathbb{R}^n$. For the purpose of our discussion on the spectrum of L (with zero boundary data), we may assume, by adding a constant if necessary, that $c(x) \geq 0$. Since L is not necessarily self-adjoint, its eigenvalues are in general complex numbers. However, there exists the following analog of Perron's theorem for positive matrices.

Theorem 1.1.

(i) *There exists a real eigenvalue $\lambda_1 > 0$ for the operator L with zero boundary condition.*

(ii) *This eigenvalue is* **of multiplicity one**, *in the sense that there exists an eigenfunction $w_1(x) > 0$ in Ω with $w_1|_{\partial\Omega} = 0$, and if u is any other function not*

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identically zero satisfying $Lu = \lambda_1 u$, in Ω and $u|_{\partial\Omega} = 0$, then u must be a multiple of w_1.

(iii) *Furthermore, if for some function v not identically zero, $Lv = \lambda v$ and $\lambda \neq \lambda_1$, then $\Re e(\lambda) > \lambda_1$.*

The eigenvalue λ_1 above is called the **principal eigenvalue** of the operator L. We can find a proof of Theorem 1.1 in [**2**, Section 6.5], which is now a classic text on partial differential equations (PDE). The proof in [**2**] makes use of iterations and the sophisticated Schaefer's fixed point theorem. In this note, we begin by giving a simple proof of Perron's original theorem for positive matrices. Then using the idea of this proof, together with some standard results in the basic theory of second-order elliptic PDE, such as the strong maximum principle, we give a more direct proof of Theorem 1.1.

The main property of Sobolev spaces $H^k(\Omega) = W^{k,2}(\Omega)$ that will be needed here is the Rellich–Kondrachov compactness theorem. Our proof (as well as that of [**2**]) also assumes some knowledge of the L^2-theory of elliptic operators concerning the solvability and the regularity of weak solutions. We can find these basic results, for example, in Theorem 1 of Section 5.7, Theorem 6 of Section 6.2, and Theorem 5 of Section 6.3 in [**2**].

2. A PROOF OF PERRON'S THEOREM. In this section, we give a proof of Perron's theorem. In the next section, we will use analogous methods for the proof of the corresponding PDE result. We first fix the convention that for an $(m \times n)$-matrix $B = (\beta_{ij})$, $B > 0$ (respectively, $B \geq 0$) means that each entry $\beta_{ij} > 0$ (respectively, $\beta_{ij} \geq 0$), and $A \geq B$ means that $A - B \geq 0$. For a vector $\mathbf{x}$, we apply the same convention by viewing it as an $(m \times 1)$-matrix. Below is a proof of Perron's theorem for a positive square $(m \times m)$-matrix A.

Proof. Set $\Lambda = \{\varepsilon > 0 | \varepsilon \in \mathbb{R},\ A\mathbf{x} \geq \varepsilon\mathbf{x},\ \text{for some vector } \mathbf{x} \geq \mathbf{0}, \mathbf{x} \neq \mathbf{0}\}$.

It is easy to see that $\Lambda \neq \emptyset$ and that it is bounded. Let $\lambda_1 = \sup \Lambda$. We now show that there exists a vector $\mathbf{x} > \mathbf{0}$ such that $A\mathbf{x} = \lambda_1 \mathbf{x}$. First, by the definition of λ_1, we can pick $\lambda^{(j)} \in \Lambda$ such that the sequence $\{\lambda^{(j)}\}$ converges to λ_1. By the definition of Λ we also have vectors $\mathbf{x}^{(j)} \geq 0$ such that $A\mathbf{x}^{(j)} \geq \lambda^{(j)}\mathbf{x}^{(j)}$. Without loss of generality, we may choose $\mathbf{x}^{(j)}$ with $\|\mathbf{x}^{(j)}\| = 1$. After possibly passing to a subsequence, we may also assume that $\mathbf{x}^{(j)}$ converges to a vector $\mathbf{x}$. By the way $\mathbf{x}$ is obtained, it is clear that $\mathbf{x} \geq \mathbf{0}$, $A\mathbf{x} \geq \lambda_1 \mathbf{x}$, and $\|\mathbf{x}\| = 1$. To show that $A\mathbf{x} = \lambda_1 \mathbf{x}$, we use the following simple lemma. ■

Lemma 2.1. *For any vector $\mathbf{y} \geq \mathbf{0}$ with $\mathbf{y} \neq \mathbf{0}$, $A\mathbf{y} > \mathbf{0}$. In particular, for any real vector $\mathbf{z}$, there exists real number $\varepsilon > 0$ such that $A\mathbf{y} > \varepsilon\mathbf{z}$.*

Proof. The assumption $\mathbf{y} \neq \mathbf{0}$ implies that $y_j > 0$ for some $1 \leq j \leq m$. Hence, $\sum_{k=1}^{m} a_{ik} y_k \geq a_{ij} y_j > 0$. For the second statement, we may let, for example,

$$\varepsilon = \frac{\delta}{\max_{1 \leq i \leq m}(|z_i| + 1)}$$

where $\delta = \min_{1 \leq i \leq m}(A\mathbf{y})_i$. ■

To finish the proof of $A\mathbf{x} = \lambda_1 \mathbf{x}$, we use reductio ad absurdum. Assume that $A\mathbf{x} \neq \lambda_1 \mathbf{x}$ and we will derive a contradiction. Let $\mathbf{y} = A\mathbf{x} - \lambda_1 \mathbf{x}$. Thus $\mathbf{y} \neq \mathbf{0}$ by assumption, and since $\mathbf{y} = \lim_{i\to\infty} A\mathbf{x}^{(j)} - \lambda^{(j)}\mathbf{x}^{(j)}$, we also have $\mathbf{y} \geq \mathbf{0}$. By Lemma 2.1, there is an $\varepsilon > 0$ so that $A\mathbf{y} > \varepsilon A\mathbf{x}$. This implies that $A\mathbf{z} > (\lambda_1 + \varepsilon)\mathbf{z}$ for $\mathbf{z} = A\mathbf{x} > \mathbf{0}$, contradicting the fact that $\lambda_1 = \sup \Lambda$. Hence, $A\mathbf{x} = \lambda_1\mathbf{x}$. In addition, Lemma 2.1 implies that $A\mathbf{x} > 0$ and also $\mathbf{x} > \mathbf{0}$, i.e., all the entries of the eigenvector $\mathbf{x}$ are positive.

To finish the proof of Perron's theorem, we must show that, up to a positive scalar constant, $\mathbf{x}$ is the unique eigenvector with eigenvalue $\lambda_1 > 0$. First, we observe that if $\mathbf{x}' \neq \mathbf{0}$ is a vector such that $A\mathbf{x}' = \lambda_1\mathbf{x}'$, and $\mathbf{x}' \geq \mathbf{0}$, Lemma 2.1 implies $\mathbf{x}' > \mathbf{0}$. Now for any vector $\mathbf{y} \neq \mathbf{0}$ with $A\mathbf{y} = \lambda_1\mathbf{y}$, we can find a real number c such that $c\mathbf{x} - \mathbf{y} \geq \mathbf{0}$ and such that at least one entry of $c\mathbf{x} - \mathbf{y}$ is equal to 0 (i.e., there exists i, with $1 \leq i \leq m$ so that $cx_i - y_i = 0$). We claim that this implies $c\mathbf{x} - \mathbf{y} = \mathbf{0}$ so that $\mathbf{y} = c\mathbf{x}$. Suppose not; then if we let $\mathbf{x}' = c\mathbf{x} - \mathbf{y}$, we would have $\mathbf{x}' \neq \mathbf{0}$, $A\mathbf{x}' = \lambda_1\mathbf{x}'$, and $\mathbf{x}' \geq 0$, but $\mathbf{x}'$ is not positive, contradicting the observation we made at the beginning of this paragraph. This proves that $\mathbf{y} = c\mathbf{x}$ and λ_1 is of multiplicity one. ■

We now make an additional observation. If λ is an eigenvalue (which in general is a complex number) of A with an eigenvector $\mathbf{z}$, then let $\mathbf{w} = \text{abs}(\mathbf{z})$, the nonnegative vector obtained by taking the norm of each entry of the vector $\mathbf{z}$. It is easy to see that $A\mathbf{w} \geq |\lambda|\mathbf{w}$, and equality holds if and only if $\mathbf{w} > 0$, $\lambda > 0$. Since $\lambda_1 = \sup \Lambda$, it implies that $\lambda_1 \geq |\lambda|$.

The above proof is basically the same as that of [**1**], which was attributed to Bohnenbust.

3. THE PDE CASE. We proceed to give a proof of Theorem 1.1 along the same line of arguments as above. Let L be the uniformly elliptic operator of Section 1. First, recall the following strong maximum principle (see Corollary 2.8, 2.9 of [**4**], as well as Theorem 4 and Lemma of Section 6.4 in [**2**]).

Theorem 3.1. *Assume that $u \in C^2(\Omega) \cap C(\overline{\Omega})$ satisfies $Lu \geq 0$ and $u|_{\partial\Omega} = 0$. Then $u > 0$ in Ω unless $u \equiv 0$. If u is not identically 0, then $\frac{\partial u}{\partial \nu} < 0$ on $\partial\Omega$, where ν is the exterior unit normal of $\partial\Omega$.*

As a consequence of the above maximum principle, we conclude that 0 is not an eigenvalue of L. Hence, by [**2**, Theorem 6 of Ch. 6.2], L has a well-defined inverse L^{-1} on $L^2(\Omega)$ such that it is a bounded operator from $L^2(\Omega) \to H^2(\Omega) \cap H_0^1(\Omega)$, where $H_0^1(\Omega) = W_0^{1,2}(\Omega)$ denotes the first L^2-Sobolev space with vanishing boundary value. This implies that there exists C such that

$$\|L^{-1}(f)\|_2 \leq C\|f\|_{L^2}. \tag{2}$$

Here $\|\cdot\|_k$ denotes the Sobolev norm of $H^k(\Omega)$. Moreover, by [**2**, Theorem 5 of Ch. 6.3] there exists $C_k \geq 0$ such that $\|L^{-1}(f)\|_{k+2} \leq C_k\|f\|_k$ for any $k \geq 0$. Let k_0 be an integer so large, but fixed, that $C^2(\overline{\Omega}) \subset H^{k_0-1}(\Omega)$, and let X be the Hilbert space $H^{k_0}(\Omega) \cap H_0^1(\Omega)$. Elliptic regularity ensures that L^{-1} maps X into X. In our discussions below, $L^{-1} : X \to X$ is the infinite dimensional analogue of the linear transformation defined by a positive square matrix $A : \mathbb{R}^m \to \mathbb{R}^m$ in the previous section.

For the proof of Theorem 1.1, we need an infinite dimensional analogue of Lemma 2.1.

Lemma 3.1. *For any nonzero $u \in X$, $u \geq 0$, let $w = L^{-1}(u)$. Then $w > 0$ in Ω and $w|_{\partial\Omega} = 0$. Furthermore, for any $v \in C^2(\overline{\Omega})$ with $v|_{\partial\Omega} = 0$, there exists an $\varepsilon > 0$ such that $w \geq \varepsilon v$.*

Proof. By the maximum principle, Theorem 3.1, we conclude that $w > 0$ in Ω and $\frac{\partial w}{\partial \nu} < 0$ on $\partial\Omega$. We now prove the last statement. Consider a general boundary point $x \in \partial\Omega$. After a local change of coordinates, we may assume that there is a neighborhood U of x on which is defined a coordinate system $x = (x', x_n)$ with $x' = (x_1, \ldots, x_{n-1})$ such that $U \cap \partial\Omega$ is defined by $x_n = 0$ and $U \cap \Omega$ is defined by $x_n > 0$. Theorem 3.1 implies $\frac{\partial w}{\partial x_n} > 0$. Consequently, by the Taylor expansion in x_n, we have

$$w(x', x_n) = w(x', 0) + \frac{\partial w}{\partial x_n}(x', 0)x_n + o(x_n)$$

and

$$v(x', x_n) = v(x', 0) + \frac{\partial v}{\partial x_n}(x', 0)x_n + o(x_n).$$

Observe that $w(x', 0) = v(x', 0) = 0$ and $\frac{\partial w}{\partial x_n}(x', 0) > 0$. Therefore, a comparison of the two equations above shows that in a neighborhood of $\partial\Omega$, $w \geq \varepsilon v$ for some $\varepsilon > 0$. By the continuity of w, v and the positivity of w in Ω, this implies the claim of the lemma. ∎

Now we present the proof of Theorem 1.1, the infinite dimensional analogue of Perron's theorem.

Proof. Let $\widetilde{\Lambda} = \{\varepsilon > 0 | L^{-1}(f) > \varepsilon f \text{ in } \Omega, \text{ for some } f \in X, f > 0 \text{ in } \Omega\}$. In view of (2), it is easy to see that $\widetilde{\Lambda}$ is bounded. Lemma 3.1 implies that $\widetilde{\Lambda} \neq \emptyset$. Let $\mu_1 = \sup \widetilde{\Lambda}$. Pick $\mu^{(j)} \in \widetilde{\Lambda}$ with $\mu^{(j)} \to \mu_1$. By the definition of $\widetilde{\Lambda}$, there exist $u_j \in X$, with $u_j(x) > 0$ for $x \in \Omega$ and $u_j|_{\partial\Omega} = 0$ such that $L^{-1}(u_j) > \mu^{(j)} u_j$. Without loss of generality, we assume that $\|u_j\|_{L^2} = 1$. Since we can only infer that a subsequence of $\{u_j\}$ is weakly convergent, we shall employ a finite iteration to get a better convergent subsequence $\{z_j\}$ with each term z_j satisfying the same properties as the corresponding u_j.

First, for any fixed j and l with $1 \leq l \leq k_0$, let $z_{j,l} = (L^{-1})^l(u_j)$. We prove inductively using the maximum principle that $z_{j,l} > \mu^{(j)} z_{j,l-1}$ in Ω. Indeed, for $l = 1$, this follows from the choice of $z_{j,1} = L^{-1}(u_j)$ and $z_{j,0} = u_j$. Assume that the claimed inequality holds for l, namely $z_{j,l} > \mu^{(j)} z_{j,l-1}$. Since $z_{j,l+1} - \mu^{(j)} z_{j,l} = L^{-1}(z_{j,l} - \mu^{(j)} z_{j,l-1})$, Theorem 3.1 and the inductive hypothesis imply that $z_{j,l+1} > \mu^{(j)} z_{j,l}$.

On the other hand, the standard elliptic estimate ([**2**, Theorem 2, Section 6.3]) asserts that there exists a positive constant C depending on L and Ω such that $\|z_{j,l}\|_{2l} \leq C^l \|u_j\|_{L^2}$. Let $z_j = z_{j,k_0}$. In particular, there exists a constant C' independent of j such that $\|z_j\|_{2k_0} \leq C'$. By the Rellich–Kondrachov compactness theorem, after passing to a subsequence if necessary, $\lim_{j\to\infty} z_j = w_1$ in $H^{k_0}(\Omega)$-norm, for some $w_1 \in X$. Note that $L^{-1}(z_j) = z_{j,k_0+1} \geq \mu^{(j)} z_{j,k_0} = \mu^{(j)} z_j$. From this, we deduce that $L^{-1}(w_1) \geq \mu_1 w_1$, $w_1|_{\partial\Omega} = 0$, and $w_1 \geq 0$. To see that $w_1 \neq 0$, we observe that $z_j \geq (\mu^{(j)})^{k_0} u_j \geq 0$; hence, $\|z_j\|_{L^2} \geq (\mu^{(j)})^{k_0}$. Now the argument in the linear algebra proof, replacing Lemma 2.1 by Lemma 3.1, implies that $L^{-1} w_1 = \mu_1 w_1$. In particular, $w_1 > 0$ in Ω and $w_1|_{\partial\Omega} = 0$. Taking $\lambda_1 = \frac{1}{\mu_1}$, this proves the existence of the principal eigenvalue and a corresponding positive eigenfunction.

To show that λ_1 is of multiplicity one, assume that $Lu = \lambda_1 u$ for a real-valued function u. Equivalently, we have $L^{-1}u = \mu_1 u$ with $\mu_1 = \frac{1}{\lambda_1}$. Let $\Lambda' = \{\eta > 0 | w_1 - \eta u \geq 0\}$. Clearly Λ' is nonempty by Lemma 3.1 and is bounded. Then let $\eta_1 = \sup \Lambda'$.

Lemma 3.1 asserts that $w_1 - \eta_1 u \equiv 0$ since otherwise $L^{-1}(w_1 - \eta_1 u) = \mu_1(w_1 - \eta_1 u) > \varepsilon u$ for some $\varepsilon > 0$, which contradicts η_1 being the supremum of Λ'. Thus, u is a multiple of w_1. (This particular part of the argument is quite close to that of [**2**, Section 6.5].)

Finally, we prove part (iii) of Theorem 1.1. Let u (which in general is complex valued) be an eigenfunction with eigenvalue λ. By applying the maximum principle to $|u|^2/w_1^{2-2\varepsilon}$, for any small $\varepsilon > 0$, it was shown in [**2**, Section 6.5] (see also [**6**]) that $\Re e(\lambda) \geq \lambda_1$. This result can also be proved via the following slightly different argument, which also shows a sharper result: $\Re e(\lambda) > \lambda_1$, for any $\lambda \neq \lambda_1$. That is a sharpening of the result.

Let $v = \frac{u}{w_1}$, $L' = L - c(x)$. A direct calculation yields

$$L'|v|^2 = 2(\Re e(\lambda) - \lambda_1)|v|^2 - 2\sum_{i,j=1}^{n} a^{ij} \frac{\partial v}{\partial x_i}\frac{\partial \bar{v}}{\partial x_j} + 2\sum_{i,j=1}^{n} a^{ij} \frac{\partial \log w_1}{\partial x_i}\frac{\partial |v|^2}{\partial x_j}.$$

As in [**7**], the regularity of $|v|^2$ (which implies the finiteness of $L'|v|^2$) and the singularity of $\frac{\partial \log w_1}{\partial x_i}$ along $\partial\Omega$ imply that $\frac{\partial |v|^2}{\partial N} = 0$ on $\partial\Omega$. Here $\frac{\partial |v|^2}{\partial N}$ is defined as $\sum_{i,j=1}^{n} a^{ij} \frac{\partial |v|^2}{\partial x_i} \nu_j$ with ν being the exterior normal and $\nu_j = \langle \nu, \frac{\partial}{\partial x_j} \rangle$. Suppose that $\Re e(\lambda) \leq \lambda_1$. Since v is nonconstant, the strong maximum principle implies that $|v|^2$ can only attain its maximum (on $\overline{\Omega}$) at some boundary point $x_0 \in \partial\Omega$. But Hopf's lemma (see Theorem 2.5 of [**4**]) asserts that $\frac{\partial |v|^2}{\partial N} > 0$ at x_0, which contradicts the just-established conclusion that $\frac{\partial |v|^2}{\partial N} = 0$ on $\partial\Omega$. The contradiction proves $\Re e(\lambda) > \lambda_1$. ■

Note that the same argument as the one above proves that if μ is any *Neumann eigenvalue* of the operator L with nonconstant eigenfunction, then $\Re e(\mu) > 0$. In fact, the slightly better result $\Re e(\mu) > \min_{\Omega} c(x)$ holds.

It seems interesting to estimate the gap $\Re e(\lambda) - \lambda_1$ from below in terms of the *geometry* of the coefficients $a^{ij}(x)$, $b^k(x)$, and $c(x)$, as well as that of Ω. Another interesting question is whether there is a generalization of this result to hypo-elliptic operators. Similarly, we can ask for an effective positive lower estimate for the nontrivial Neumann eigenvalue μ (i.e., the eigenvalue with nonconstant eigenfunctions) for the operator L with $c(x) = 0$. For this last problem, consult [**5**] for some recent progress for the case that the domain Ω is convex.

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