The Poisson Equation and Hermitian-Einstein Metrics on Holomorphic Vector Bundles over Complete Noncompact Kähler Manifolds

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1. INTRODUCTION

The existence of Hermitian-Einstein metrics on compact Kähler manifolds has been well-understood after the work of [9] and [27]. In [20] we studied it on complete noncompact Kähler manifolds which have positive lower bound on the spectrum of the Laplacian operator. In [21], it was shown that there is a relation between solving the Poisson equation and the Poincaré-Lelong equation. In this paper we shall illustrate the relation between solving the Poisson equation (heat equation) and the Hermitian-Einstein equation (Hermitian-Einstein heat equation). The general idea is that by first solving the Hermitian-Einstein equation on the determinant line bundle, which is a Poisson equation, together with some desirable estimates, one can use the solution as a barrier to construct a solution to the corresponding nonlinear equation on the vector bundle. (In [22], we will show that there is a similar relation between the Poisson equation and the Kähler-Ricci flow equation on the complete Kähler manifolds with bounded nonnegative bisectional curvature. There, the authors use the solution to the Poisson equation to reduce the curvature estimate, therefore the long time existence, to a simple use of the maximum principle of Karp-Li on complete Riemannian manifolds, which considerably simplifies the previous work of Shi [24]-[25].) By exploring this idea we give a simple and unified proof on the existence of Hermitian-Einstein metrics on complete Kähler manifolds under various assumptions. The main results can be found in Theorems 2.9 and 2.10. There is an analogy between the Kähler-Einstein metrics on the tangent bundle and the Hermitian-Einstein metrics on general holomorphic vector bundles. The previous proved result in [20] corresponds to the existence result of Cheng-Yau [7] on the Kähler-Einstein metrics on bounded strictly pseudoconvex domains in C^m. The results in Theorems 2.9 and

2.10 of this paper are more in the spirit of the results proved by Bando-Kobayashi [2], Joyce [13] and Tian-Yau [26] on the existence of Ricci flat Kähler metrics. The various Sobolev inequalities play an important role in both our proof and the previous constructions in [26].

In the second part of this paper we study the Laplacian heat equation (we will just call it heat equation) and its relation to the Hermitian-Einstein heat equation. For the results on the heat equation we focus on the case when the manifold M is a complete Riemannian manifold with nonnegative Ricci curvature. (There are similar results for the complete Riemannian manifolds with L^2-Sobolev inequality. See, for example, Theorem 3.6.) In Section 3 we first derive some moment type estimates (after Nash's [18]) for the heat equation (cf. Theorem 3.1). The estimates follow directly from some general results of Li-Yau [17] on the heat kernels. The derivation is more direct and simpler than Nash's original argument for the Euclidean case, which depends on an entropy estimate. The relation between the asymptotic spatial behavior of the initial data and the long time behavior of the heat equation can then be established after the moment type estimates. Among others we show the following result.

Let f be a nonnegative continuous function on M. Consider the heat equation

$$\begin{cases} \left(\Delta - \dfrac{\partial}{\partial t}\right) u(x,t) = 0, \\ u(x,0) = f(x). \end{cases}$$

Let $u(x,t)$ be its nonnegative solution. Then

(i) $\int_0^\infty u(x,s)\,ds < \infty$ *if and only if* $\int_0^\infty r k(x,r)\,dr < \infty$;
(ii) $\lim_{t\to\infty} u(x,t) = 0$ *if and only if* $\lim_{r\to\infty} k(x,r) = 0$;
(iii) $u(x,t) = O(\log(t+1))$ *if and only if* $k(x,r) = O(\log(r+1))$;
(iv) $u(x,t) = O(t^d)$ *if and only if* $k(x,r) = O(r^{2d})$, *for any d with* $-n/2 - 1 < d < \infty$.

Here as in [21]

$$k(x,r) = \frac{1}{V_x(r)} \int_{B_x(r)} f(y)\,dv_y.$$

The above statement can be thought as a parabolic analogue of the results proved in [21]. In particular, the second part of the above statement and Corollary 3.3 give partial generalizations of a result of Li [14] on the large time behavior of the heat equation, since we assume neither that the manifold is of maximum volume growth nor the boundedness of the initial data. In [22], surprisingly, we find a similar moment type estimate for the Kähler-Ricci flow. It turns out that one side of the moment estimate is nothing but an improved version of the volume element estimates obtained by Shi in [Sh1-2]. The derivation for the linear case here also suggests a simple proof for the other side of the moment estimate for the nonlinear Kähler-Ricci flow, which was discovered in [6] very recently. This unified treatment certainly clarifies and simplifies the previous results on Kähler-Ricci

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flow. Using the results on the heat equation in Section 3, we study the Hermitian-Einstein heat equation in Section 4. The main result is contained in Theorem 4.1 and Theorem 4.6. In Theorem 4.1 we prove a long time existence of the Hermitian-Einstein heat equation on any complete Kähler manifold, under the assumption that the initial metric has bounded mean curvature. In Theorem 4.6 we prove the convergence on a complete Ricci nonnegative manifold under some average decay assumptions of the initial metric's mean curvature, which gives a heat flow proof of Theorem 2.10. Under an assumption similar to that of Theorem 4.6, in [24] and [25], W.X. Shi proved the long time existence of the Ricci flow equation on a complete Kähler manifold with nonnegative bisectional curvature. Our assumption here is slightly stronger than his. However, we should point out that the stronger assumption is used to prove the convergence of the Hermitian-Einstein flow, and we have shown the long time existence under much more flexible condition in Theorem 4.1. On the other hand, a simple use of H. D Cao's Li-Yau-Hamilton inequality (cf. [4]) shows that, for Kähler-Ricci flow equation, the solution never converges except that the initial metric is flat (cf. [22, Corollary 2.3]). Finally we should point out that the method developed here can be modified to construct harmonic maps from various complete noncompact Riemannian manifolds to Riemannian manifolds with nonpositive sectional curvature or a domain in a manifold with a convex defining function.

2. Hermitian-Einstein equation

In this section we prove the existence of Hermitian-Einstein metrics on a holomorphic vector bundle E over a complete Kähler manifold M, assuming that E admits a Hermitian metric H_0 such that the mean curvature of H_0 satisfies certain decay conditions. This assumption has appeared in various ocassions dealing with the nonlinear elliptic or parabolic equations, such as the harmonic map equation and Ricci flow on complete noncompact manifolds. We focus on the elliptic method here and deal with the corresponding parabolic equation later. The main result is that we can solve the Hermitian-Einstein equations, a nonlinear system, when the corresponding linear equations on the determinant line bundle have a nonnegative solution. In [20], the existence result has been proved for the manifold with $\lambda_1(M) > 0$, where $\lambda_1(M)$ denotes the lower bound of the spectrum of the Laplacian operator. Here we are going to prove it for a class of Kähler manifolds, including ones with nonnegative Ricci curvature. The method we use is much simpler than the previous one. It reduces the global nonlinear problem to some local a priori estimates and solving a global linear problem with nonnegative solution.

First let us fix the notation. Let M^m be a Kähler manifold of complex dimension m, and let (E, H) be a holomorphic vector bundle with a Hermitian metric H. Denote by $\omega = \sqrt{-1} g_{\alpha\bar{\beta}} dz^\alpha \wedge d\bar{z}^\beta$ the Kähler form, where $g = g_{\alpha\bar{\beta}} dz^\alpha \otimes d\bar{z}^\beta$ is the Kähler metric. Then we can define the operator Λ

as the contraction with $-\sqrt{-1}\omega$, i.e., for any $(1,1)$-form $\sum a_{\alpha\bar{\beta}} dz^\alpha \wedge d\bar{z}^\beta$

$$\Lambda \sum a_{\alpha\bar{\beta}} dz^\alpha \wedge d\bar{z}^\beta = \sum g^{\alpha\bar{\beta}} a_{\alpha\bar{\beta}}.$$

A connection A on the vector bundle E is called *Hermitian-Einstein* (cf. [9]) if the corresponding curvature form F_A satisfies the following equation:

$$\Lambda F_A = \lambda I, \tag{2.1}$$

where λ is some constant. In the case when M is a compact Kähler manifold λ is a topological invariant (cf. [27]). If for a Hermitian metric H on E, the corresponding Hermitian connection is Hermitian-Einstein, then the metric H is called a *Hermitian-Einstein metric* (cf. [9]). Recall that under a holomorphic local frame $\{e_i\}$, if $H = (h_{i\bar{j}})$, where $h_{i\bar{j}} = \langle e_i, e_j \rangle$, then the Hermitian connection A_H and the corresponding curvature form F_H are given by the following formulae:

$$A_H = \partial H H^{-1}, \tag{2.2}$$

$$F_H = \bar{\partial}(\partial H H^{-1}). \tag{2.3}$$

If we write $F_H = F^j_{i\alpha\bar{\beta}} dz^\alpha \wedge d\bar{z}^\beta$, we define

$$\Lambda F_H = \sum_{\alpha,\beta} g^{\alpha\bar{\beta}} F^j_{i\alpha\bar{\beta}}$$

the *mean curvature* of H. We also denote $R(x) = \operatorname{tr} \Lambda F_H$, the *scalar curvature* of H. In this paper we will focus on the case when $\lambda = 0$. The general cases do not differ from this special case. In [20] we proved that, for the complete Kähler manifold with $\lambda_1(M) > 1$, if there is a Hermitian metric H_0 such that the mean curvature of H_0 satisfies that $\|\Lambda F_{H_0}\| \in L^p(M)$ for a $p > 1$, then there exists a metric H on E such that $\Lambda F_H \equiv 0$. A similar result was proved for Kähler-Einstein metrics before in [7]. In [20] we applied a heat flow method. Namely, we first proved that the Hermitian-Einstein flow has a global solution. Then we showed that it converges to a solution to the elliptic problem. In the following we will adapt the direct elliptic method. The elliptic method reduces the problem essentially to a linear problem. This method can also establish some existence results, for the Hermitian-Einstein metrics on holomorphic vector bundles over complete Kähler manifolds with nonnegative Ricci curvature, or general nonparabolic Kähler manifolds (see Theorems 2.9 and 2.10 for the precise statement). Let us first start with some simple lemmas on the solutions of the Poisson equation.

Lemma 2.1. *Let M be a complete Riemannian manifold. Assume that $\lambda_1(M) > 0$. Then for a nonnegative continuous function f the Poisson equation*

$$\Delta u = -f$$

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has a nonnegative solution $u \in W^{2,n}_{\mathrm{loc}}(M) \cap C^{1,\alpha}_{\mathrm{loc}}(M)$ $(0 < \alpha < 1)$ *if* $f \in L^p(M)$ *for some* $p \geq 1$.

Proof. Let Ω_i be a compact exhaustion of M. We first solve the following Dirichlet problem on Ω_i:

$$\begin{cases} \Delta u_i = -f, \\ u_i|_{\partial\Omega_i} = 0. \end{cases} \tag{2.4}$$

First, by the maximum principle, we know that $u_i \geq 0$. Now multiplying u_i^{p-1} on both sides of the equation and integrating by parts we have that

$$-(p-1)\int_{\Omega_i} |\nabla u_i|^2 u_i^{p-2}\, dv = \int_{\Omega_i} -f u_i^{p-1}\, dv.$$

Rewrite it as

$$\frac{4(p-1)}{p^2}\int_{\Omega_i} |\nabla (u_i)^{p/2}|^2\, dv = \int_{\Omega_i} f u_i^{p-1}\, dv.$$

Using the fact that $\lambda_1(M) > 0$, we have

$$\lambda_1(M)\int_{\Omega_i} u_i^p\, dv \leq \int_{\Omega_i} |\nabla (u_i)^{p/2}|^2\, dv.$$

Therefore we have

$$\begin{aligned} \int_{\Omega_i} u_i^p\, dv &\leq \frac{p^2}{4\lambda_1(M)(p-1)}\int_{\Omega_i} f u_i^{p-1}\, dv \\ &\leq \frac{p^2}{4\lambda_1(M)(p-1)}\Big(\int_M f^p\, dv\Big)^{1/p}\Big(\int_{\Omega_i} u_i^p\, dv\Big)^{(p-1)/p} \end{aligned}$$

from which we have

$$\Big(\int_{\Omega_i} u_i^p\, dv\Big)^{1/p} \leq \frac{p^2}{4\lambda_1(M)(p-1)}\Big(\int_M f^p\, dv\Big)^{1/p}.$$

Using the interior L^p estimates for the linear elliptic PDE (cf. [11, Theorem 9.11]) we know that, over a compact subset Ω, there will be a uniform upper bound for $\|u_i\|_{W^{2,p}(\Omega)}$. Therefore, using Rellich's compactness theorem, by passing to a subsequence we know that u_i will converge to a solution $u \in W^{2,p}_{\mathrm{loc}}(M)$ on the manifold M. Clearly $u \geq 0$, a.e. Since $f \in L^q_{\mathrm{loc}}(M)$ for any $q \geq n$, the regularity theory for the linear elliptic PDE (cf. [1, Theorem 3.54].) implies that $u \in W^{2,q}_{\mathrm{loc}}(M)$. This completes the proof of the lemma. □

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The proof above does not work for the case $p = 1$, which needs a different treatment, which we will give a little later. If on M we have the L^2-Sobolev

$$\int_M |\nabla \varphi|^2 \, dv \geq S(M) \Big(\int_M \varphi^{2n/(n-2)} \, dv \Big)^{(n-2)/n}, \tag{2.5}$$

an argument similar to the proof of Lemma 2.1 shows a similar lemma.

Lemma 2.2. *Let M be a complete Riemannian manifold satisfying the L^2-Sobolev inequality* (2.5). *Then for a nonnegative continuous function f the Poisson equation*

$$\Delta u = -f$$

has a nonnegative solution $u \in W^{2,n}_{\mathrm{loc}}(M) \cap C^{1,\alpha}_{\mathrm{loc}}(M)$ $(0 < \alpha < 1)$ if $f \in L^p(M)$ for some $n/2 > p \geq 1$.

Proof. We only need to replace the Poincaré inequality by the Sobolev inequality in the proof of Lemma 2.1. □

Remark. For $p > 1$, Lemmas 2.1 and 2.2 were essentially proved before in author's thesis [19] (cf. Theorems 2.8 and 2.9 therein).

Now we prove Lemma 2.1 and Lemma 2.2 for the case $p = 1$. The following statement was essentially proved in the proof of Theorem 3.2 in [21].

Lemma 2.3. *Let M be a complete nonparabolic manifold* (*in the sense that M supports a positive Green's function*). *Then for a nonnegative continuous function f the Poisson equation*

$$\Delta u = -f$$

has a nonnegative solution $u \in W^{2,n}_{\mathrm{loc}}(M) \cap C^{1,\alpha}_{\mathrm{loc}}(M)$ $(0 < \alpha < 1)$ if $f \in L^1(M)$.

Proof. Since M is nonparabolic, there exists a positive Green's function on M. It is sufficient if we can show that $\int_M G(x,y) f(y) \, dy$ is locally bounded. For any x, there are R such that $x \in B_o(R)$. Here o is a fixed point. Thus there are constants C, C', C'', depending only on R and the manifold M, such that

$$\begin{aligned}
\int_M G(x,y) f(y) \, dy &= \Big(\int_{B_o(2R)} + \int_{M \setminus B_o(2R)} \Big) G(x,y) f(y) \, dy \\
&\leq C + \int_{M \setminus B_o(2R)} G(x,y) f(y) \, dy \\
&\leq C + C' \int_{M \setminus B_o(2R)} G(o,y) f(y) \, dy \\
&\leq C + C' \int_{M \setminus B_o(2R)} f(y) \, dy \leq C''.
\end{aligned}$$

Here in the second last inequality we used the Harnack inequality, and in the last inequality we used the fact that $\sup_{M \setminus B_o(2R)} G(o,y) < \infty$ and the fact that f is integrable. Thus we have proved Lemma 2.3. □

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On the other hand, it is not hard to prove that M is nonparabolic under the assumption of Lemma 2.1 or Lemma 2.2. For example, the existence of the positive Green's function was proved in [23] under the assumption that $\lambda_1(M) > 0$ (cf. [23, Theorem A.3]). Under the assumption of Lemma 2.2, it was first proved by J. Nash in [18] (see also [5]) that $H(x,x,t)$, where $H(x,y,t)$ is the heat kernel of M, has an upper bound

$$H(x,x,t) \le C(M)t^{-n}.$$

By [8] (See also [12]) one has a Gaussian bound

$$H(x,y,t) \le \frac{C(M)}{t^n} \exp\left(-\frac{Cr^2(x,y)}{t}\right).$$

Integrating the above inequality along the time direction gives a positive Green's function on M. Therefore the proof for the case $p = 1$ of Lemma 2.1 and Lemma 2.2 follows from Lemma 2.3.

The next lemma deals with the case when M is a complete Riemannian manifold with nonnegative Ricci curvature.

Lemma 2.4. *Let M be a complete Riemannian manifold with nonnegative Ricci curvature. For any nonnegative continuous function $f \ge 0$, the Poisson equation*

$$\Delta u = -f$$

has a nonnegative solution (which can be chosen to be $W^{2,n}_{\mathrm{loc}}(M) \cap C^{1,\alpha}_{\mathrm{loc}}(M)$) if and only if, for every $x \in M$,

$$\int_0^\infty \frac{t}{V_x(t)} \int_{B_x(t)} f(y)\, dy\, dt < \infty.$$

Proof. This lemma was essentially proved in [21]. For the sake of completeness, we include a proof here.

Suppose that

$$\int_0^\infty \frac{t}{V_x(t)} \int_{B_x(t)} f(y)\, dy\, dt < \infty.$$

If $f \equiv 0$, then $u \equiv 0$ is a solution. Otherwise the condition above means that M is non-parabolic, by a theorem of Li-Yau in [17]. On the other hand, let $G(x,y)$ be the minimal positive Green's function; by Li-Yau's estimate (cf. [17, Theorem

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5.2]) and Fubini's theorem we have

$$\begin{aligned}\int_M G(x,y)f(y) &\le C\int_0^\infty \left(\int_r^\infty \frac{t}{V_x(t)}\,dt\right)\int_{\partial B_x(r)} f\,dA\,dr\\ &= C\int_0^\infty \left(\int_0^t \left(\frac{t}{V_x(t)}\right)\int_{\partial B_x(r)} f\,dA\,dr\right)dt\\ &= C\int_0^\infty \frac{t}{V_x(t)}\int_{B_x(t)} f(y)\,dy\,dt < \infty.\end{aligned}$$

From here it is clear to see that

$$\int_M G(x,y)f(y)\,dy$$

is a nonnegative solution to $\Delta u = -f$.

Conversely, suppose that there is a nonnegative solution to $\Delta u = f$; we need to derive the desired average decay for f. If M is a parabolic, then u is a constant. Therefore $f = 0$ and the estimate holds trivially. Otherwise we assume that M is nonparabolic. For $R > 0$, let $G_R(x,y)$ be the Dirichlet Green's function on $B_x(R)$. Then we have

$$\begin{aligned}u(x) &= -\int_{B_x(R)} G_R(x,y)\Delta u(y)\,dy - \int_{\partial B_x(R)} u\frac{\partial G_R}{\partial \nu}\\ &= \int_{B_x(R)} G_R(x,y)f(y)\,dy - \int_{\partial B_x(R)} u\frac{\partial G_R}{\partial \nu}\\ &\ge \int_{B_x(R)} G_R(x,y)f(y)\,dy.\end{aligned}$$

Letting $R \to \infty$ we have

$$\int_M G(x,y)f(y)\,dy \le u(x).$$

Then for any $R > 0$ we have

$$\begin{aligned}u(x) &\ge \int_{B_x(R)} G(x,y)f(y)\,dy\\ &\ge C(n)\int_0^R \left(\int_t^\infty \frac{s}{V_x(s)}\,ds\int_{\partial B_x(t)} f(y)\,dA\right)dt\\ &= C(n)\left(\left(\int_R^\infty \frac{t}{V_x(t)}\,dt\right)\int_{B_x(R)} f(y)\,dy\right.\\ &\qquad\left. + \int_0^R \left(\frac{t}{V_x(t)}\int_{B_x(t)} f(y)\,dy\right)dt\right)\end{aligned}$$

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$$\geq C(n)\int_0^R \left(\frac{t}{V_x(t)}\int_{B_x(t)} f(y)\,dy\right)dt.$$

Since R is arbitrary we finish the proof of the lemma. □

Now we use the method of compact exhaustion to prove the existence of the Hermitian-Einstein metrics. The argument shown below gives another proof of the result in [20], using Lemma 2.1.

By [10], we know that the following Dirichlet problem is solvable on Ω_i. Namely, there exists a Hermitian metric $H_i(x)$ such that

$$(D_i)\qquad \begin{cases} \Lambda F_{H_i} = 0, & \text{for } x \in \Omega_i, \\ H_i(x)\,|_{\partial\Omega_i} = H_0(x). \end{cases}$$

In order to prove that we can pass to limit and eventually obtain a solution on the whole manifold M, we need to establish some a priori estimates. The key is the so-called zeroth order estimate. In order to do that we have to recall some functions defined by Donaldson. The first one is

$$\tau_i(x) = \tau(H_i, H_0) = \mathrm{tr}(H_i H_0^{-1}).$$

Since $H_i H_0^{-1}$ is a section of $\mathrm{Hom}(E,E)$, it is a well-defined function. The second one is

$$\sigma_i(x) = \sigma(H_i, H_0) = \mathrm{tr}(H_i H_0^{-1}) + \mathrm{tr}(H_0 H_i^{-1}) - 2k,$$

where k is the rank of E. $\sigma(x)$ measures the "distance" of two metrics in the C^0 topology, in the sense that $H_i \to H_0$ if $\sigma(H_i, H_0) \to 0$ uniformly.

A Bochner type inequality is the following one. The proof is similar to that of Proposition 3.7 in [20].

Lemma 2.5. *Let $\tau_i(x)$, H_0 and H_i as above. Then*

$$(2.6)\qquad \begin{cases} \Delta \log \tau_i \geq -\|\Lambda F_{H_0}\| & \text{for } x \in \Omega_i, \\ \log \tau_i\,|_{\partial\Omega_i} = \log k. \end{cases}$$

Let $f_i = \log \tau_i - \log k$. We then have

$$(2.7)\qquad \begin{cases} \Delta \log f_i \geq -\|\Lambda F_{H_0}\| & \text{for } x \in \Omega_i, \\ \log f_i\,|_{\partial\Omega_i} = 0. \end{cases}$$

Lemma 2.6. *Assume that $\|\Lambda F_{H_0}\| \in L^p(M)$ for some $p \geq 1$ and g is the solution to the Poisson equation $\Delta g = -\|\Lambda F_{H_0}\|$ constructed by Lemma 2.1. Then we have*

$$f_i \leq g \quad \text{or} \quad \tau_i \leq k e^g.$$

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Proof. Noticing that $f_i - g$ is subharmonic and $f_i - g|_{\partial\Omega_i} \le 0$, the proof is just the maximum principle (cf. [11, Theorem 9.1]).

Since the same Bochner type inequality holds for $\log \mathrm{tr}(H_0 H_i^{-1})$, the same argument as the above gives the estimate for $\mathrm{tr}(H_0 H_i^{-1})$. Combining them we have the C^0-estimates. □

Proposition 2.7. *Let M be a complete Kähler manifold such that $\lambda_1(M) > 0$, and let E be a holomorphic vector bundle of* rank k *over M with metric H_0 such that $\|\Lambda F_{H_0}\| \in L^p(M)$ for some $p \ge 1$. Let H_i be the solutions to the Dirichlet problem for the Hermitian-Einstein metrics. Let g be the solution to the Poisson equation $\Delta g = -\|\Lambda F_{H_0}\|$ constructed by Lemma 2.1. Then*

$$\sigma_i \le 2ke^g - 2k.$$

Here $\sigma_i(x) = \mathrm{tr}(H_i H_0^{-1}) + \mathrm{tr}(H_0 H_i^{-1}) - 2k$.

One can see easily that the corresponding result holds if M satisfies the assumptions in Lemmas 2.2, 2.3. When M is a complete manifold with nonnegative Ricci curvature, using Lemma 2.4 and the exact same proof of the above proposition we can have the following result.

Proposition 2.8. *Let M be a complete Kähler manifold with nonnegative Ricci curvature, and let E be a holomorphic vector bundle of* rank k *over M with metric H_0 such that $\|\Lambda F_{H_0}\|$ satisfies, for every $x \in M$,*

$$\int_0^\infty \frac{t}{V_x(t)} \int_{B_x(t)} \|\Lambda F_{H_0}\|(y)\, dy < \infty.$$

Let H_i be the solutions to the Dirichlet problem for the Hermitian-Einstein metrics. Let g be the solution to the Poisson equation $\Delta g = -\|\Lambda F_{H_0}\|$ constructed by Lemma 2.4. Then

$$\sigma_i \le 2ke^g - 2k.$$

Here $\sigma_i(x) = \mathrm{tr}(H_i H_0^{-1}) + \mathrm{tr}(H_0 H_i^{-1}) - 2k$.

Once we have established the C^0-estimate, we can prove two existence theorems. The key ingredient of the proof is to obtain a C^1-estimate, which we will establish later in the proof. The case when $\lambda_1(M) > 0$ and $p > 1$ has been proved in [20] by the heat equation method.

Theorem 2.9. *Let M be a complete Kähler manifold, and let E be a holomorphic vector bundle of* rank k *over M with metric H_0 such that $\|\Lambda F_{H_0}\| \in L^p(M)$. Assume that either*

(i) *$\lambda_1(M) > 0$ and $p \ge 1$, or*
(ii) *M satisfies L^2-Sobolev and $n/2 > p \ge 1$, or*
(iii) *M is nonparabolic and $p = 1$.*

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Then there exists a metric H on E such that

$$\Lambda F_H \equiv 0.$$

Furthermore,

$$\sigma(H(x), H_0(x)) \leq 2ke^g - 2k,$$

where g is the solution to the Poisson equation $\Delta g = -\|\Lambda F_{H_0}\|$ constructed by Lemmas 2.1-2.3.

We shall only give a complete proof for the case when M has nonnegative Ricci curvature, since the argument is robust enough to apply to the other cases.

Theorem 2.10. *Let M be a complete Kähler manifold with nonnegative Ricci curvature, and let E be a holomorphic vector bundle of* rank k *over M with metric H_0 such that $\|\Lambda F_{H_0}\|$ satisfies, for every $x \in M$,*

$$\int_0^\infty \frac{t}{V_x(t)} \int_{B_x(t)} \|\Lambda F_{H_0}\|(y)\, dy < \infty.$$

Then there exists a metric H on E such that

$$\Lambda F_H \equiv 0.$$

Furthermore,

$$\sigma(H(x), H_0(x)) \leq 2ke^g - 2k,$$

where g is the solution to the Poisson equation $\Delta g = -\|\Lambda F_{H_0}\|$ constructed by Lemma 2.4.

Proof. In general for any two metrics H and K, direct calculation using (2.3) gives

$$(F_H - F_K)h = \bar\partial(\partial_K h h^{-1})h = \bar\partial\partial_K h + \partial_K h h^{-1}\bar\partial h.$$

Here $h = HK^{-1}$. If we denote $h = HH_0^{-1}$, we then can write $\Lambda F_H \equiv 0$ as

$$\Delta h = \Lambda(\partial_{H_0} h h^{-1}\bar\partial h) + \Lambda F_{H_0} h. \tag{2.8}$$

It is a semi-linear elliptic system. Solving H for the Hermitian-Einstein equation is equivalent to solving h for the above nonlinear system, where h is a smooth section of the endomorphism bundle $\mathrm{Hom}(E, E)$. Fixing H_0 is just fixing a gauge on the vector bundle. In the following discussion we will also use H_{i_0}, for some i_0, constructed before as the fixed base metric. Let Ω_i be a compact exhaustion, and let H_i be the solution to the Dirichlet problem over Ω_i. We denote $h_i = H_i H_0^{-1}$. The C^0-estimates and the proof show that for any compact subset $\Omega \subset M$ there exist positive constants C_1 and C_2 such that

$$C_2 I \leq h_i \leq C_1 I, \tag{2.9}$$

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where I is the identity on E. Here C_1 and C_2 depend only on M, Ω and k. Choose a i_0 such that $\Omega \subset \Omega_{i_0}$. In the following, we first show that, by passing to a subsequence, H_i uniformly converges to a metric H_∞, which is at least continuous, over Ω. Once we have this, the gradient estimate will follow easily.

Let $\sigma_{ij}(x) = \sigma(H_i(x), H_j(x))$. We are going to show that, by passing to a subsequence, for any $\varepsilon > 0$,

$$\int_{B_o(1)} \sigma_{jk}^2(x)\, dx \le \varepsilon, \tag{2.10}$$

for sufficiently large j and k. On the other hand, since $\Delta \log \mathrm{tr}(H_i H_j^{-1}) \ge 0$, it is not hard to verify that $\Delta\sigma_{ij} \ge 0$. Then the uniform convergence over a compact domain follows from the the uniform L^2 convergence and the mean-value inequality. Let $h_j = H_j H_{i_0}^{-1}$ and $\tau_j = \mathrm{tr}(h_j)$; a direct calculation shows that

$$\Delta\tau_j = \mathrm{tr}(\Lambda(\partial_{H_{i_0}} h_j)(h_j)^{-1}(\bar\partial h_j)). \tag{2.11}$$

Now we define $e_j(x) = \mathrm{tr}(\Lambda(\partial_{H_{i_0}} h_j)(\bar\partial h_j))$. Clearly it contains all the squares of the first order derivatives of h_j. Using (2.9) and (2.11) we have $\Delta\tau_j \ge C_3 e_j(x)$. Let φ be a cut-off function over $B_o(2)$. Multiplying $\varphi^2\tau_j$ on both sides of the above inequality and integrating by parts we have

$$\begin{aligned}
\int_{B_o(1)} e_j(x)\tau_j\, dx &\le \int_{B_o(2)} e_j(x)\tau_j\varphi^2\, dx \\
&\le \frac{1}{C_3}\int_{B_o(2)} (\Delta\tau_j)\tau_j\varphi^2\, dx \\
&= \frac{1}{C_3}\int_{B_o(2)} (-|\nabla\tau_j|^2\varphi^2 + |\nabla\varphi|\,|\nabla\tau_j|\tau_j\varphi)\, dx \\
&\le \frac{2}{C_3}\int_{B_o(2)} |\nabla\varphi|^2\tau_j^2\, dx \le \frac{2}{C_3}\int_{B_o(2)} \tau_j^2\, dx.
\end{aligned}$$

Combining with (2.9) we know that

$$\int_{B_o(1)} e_j(x)\, dx \le \frac{2}{C_1 C_3}\int_{B_o(2)} \tau_j^2\, dx.$$

Since τ_j are uniformly bounded, we know that h_j is bounded in $W^{1,2}(B_o(1))$. By the Sobolev embedding theorem we know that by passing to a subsequence it converges in $L^2(B_o(1))$. This implies (2.10). Now using the subharmonicity of σ_{jk} we have that, over a compact subdomain Ω, H_j uniformly converges to a metric $H_\infty(x)$, which is at least continuous.

In order to prove that one can extract a convergent subsequence out of H_i, we need to establish the estimates for the higher order derivatives. The equation (2.8)

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and the standard "bootstrapping" for the elliptic PDE make it sufficient to only prove a uniform estimate for the C^1 bounds. Namely, we only need to show that for a compact subset $\Omega \subset M$, there exists a constant C_4, independent of j, such that

$$\sup_{\Omega} |\nabla H_j| \leq C_4. \tag{2.12}$$

It will be enough if we can prove it for small balls. Therefore, let $\Omega = B_o(\frac{1}{2})$. By the estimate in the last paragraph we know that there exists a constant C, independent of j, such that:

$$\int_{B_0(1)} |\nabla H_j|^2 \, dx \leq C.$$

By our C^0 estimate (2.9), it implies that:

$$\int_{B_0(1)} |(\nabla H_j) H_j^{-1}|^2 \leq C'$$

for some constant $C' > 0$ independent of j. Also we can assume that $B_o(\frac{1}{2})$ is small enough such that the vector bundle is trivial over $B_o(\frac{1}{2})$. We also know that $|(\nabla H_j)H_j^{-1}|^2$ is a subharmonic function by [10] (see [10, Section 2.3]). By the mean-value inequality of Li-Schoen [15], or the local version, [11, Theorem 8.17], we have that

$$\sup_{Bo(1/2)} |(\nabla H_j) H_j^{-1}|^2 \leq C'' \int_{B_0(1)} |(\nabla H_j) H_j^{-1}|^2.$$

Here C'' only depends on the volume of $B_o(\frac{1}{2})$ and the lower bound of curvature of the Kähler metric ω on $B_o(1)$. Combining them together we have (2.12).

Once we have the C^1-estimate (2.12), the standard elliptic theory shows that, by passing to a subsequence, H_j converges uniformly over any compact subset of M to a metric H_∞ which is a solution of $\Lambda F_{H_\infty} \equiv 0$ on the whole manifold. Therefore we complete the proof of Theorem 2.10. □

3. Moment type estimates

In this section we establish some "moment" type estimates for the solution to the heat equation. The method we use here is different from Nash's method in the sense that we do not need to introduce the "entropy" quantity, and derive the estimate directly from the Guassian type upper bound of the heat kernel proved by Li-Yau [17]. The proof is also simpler. A by-product result, which partially generalizes a theorem of Li on the long time behavior of the solution to the heat equation on Ricci nonnegative manifold, follows afterwards. Let us start with the "moment" type estimate first.

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Theorem 3.1. *Let M be a Riemannian manifold with nonnegative Ricci curvature. Let $u(x,t)$ be the nonnegative solution to the heat equation*

$$\begin{cases} \left(\Delta - \dfrac{\partial}{\partial t}\right) u(x,t) = 0, \\ u(x,0) = f(x). \end{cases}$$

Here f is a nonnegative function. As in [21] *we define*

$$k(x,r) = \frac{1}{V_x(r)} \int_{B_x(r)} f(y)\, dv_y.$$

Then for any d $(-n-2 < d < \infty)$ there exists a constant $C(n,d)$ such that

$$u(x,t) \le C(n,d) A t^{d/2}, \quad \text{for } t \ge 1 \text{ and some } A > 0$$

if and only if $k(x,r) \le Ar^d$, for $r \ge 1$. Here n is the dimension of the manifold. Moreover, if $A^{-1} r^d \le k(x,r) \le Ar^d$, we have the "moment" estimate:

$$C^{-1}(n,d) A^{-1} t^{d/2} \le u(x,t) \le C(n,d) A t^{d/2}. \tag{3.1}$$

For the case $k(x,r) \le A \log(1+r)$ we have the similar estimate:

$$u(x,t) \le C(n) A \log(1+t),$$

and $u(x,t) \le A \log(1+t)$ also implies that $k(x,r) \le C(n) A \log(1+r)$.

Proof. First, by the uniqueness of the nonnegative solution, we can write $u(x,t)$ in terms of the heat kernel on M as

$$u(x,t) = \int_M H(x,y,t) f(y)\, dv_y.$$

Now recall the heat kernel estimate of Li-Yau, which says that, for some $C = C(n)$,

$$H(x,y,t) \le \frac{C(n)}{V_x(\sqrt{t})} e^{-r^2(x,y)/((4+\varepsilon)t)}.$$

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Then we have

$$\begin{aligned} u(x,t) &\le \frac{C(n)}{V_x(\sqrt{t})}\int_0^\infty e^{-r^2/((4+\varepsilon)t)}\left(\int_{\partial B_X(r)} f(y)\,dA_y\right)dr \\ &\le \frac{C(n)}{V_x(\sqrt{t})}\int_0^{\sqrt{t}}\int_{\partial B_X(r)} f(y)\,dA_y\,dr \\ &\quad + \frac{C(n)}{V_x(\sqrt{t})}\int_{\sqrt{t}}^\infty e^{-r^2/((4+\varepsilon)t)}\left(\int_{\partial B_X(r)} f(y)\,dA_y\right)dr \\ &\le C(n)At^{d/2} + C(n)\int_{\sqrt{t}}^\infty \left(\frac{1}{V_x(\sqrt{t})}\int_{B_X(r)} f(y)\,dv_y\right) \\ &\qquad\qquad \times \frac{2r}{(4+\varepsilon)t}e^{-r^2/((4+\varepsilon)t)}\,dr. \end{aligned}$$

The second term above follows after integration by parts. Using the assumption and volume comparison we then have

$$\begin{aligned} u(x,t) &\le C(n)A\left(t^{d/2} + \int_{\sqrt{t}}^\infty \left(\frac{V_x(r)}{V_x(\sqrt{t})}\right)r^d\frac{2r}{(4+\varepsilon)t}e^{-r^2/((4+\varepsilon)t)}\,dr\right) \\ &\le C(n)A\left(t^{d/2} + \int_{\sqrt{t}}^\infty \left(\frac{r}{\sqrt{t}}\right)^n r^d\frac{2r}{(4+\varepsilon)t}e^{-r^2/((4+\varepsilon)t)}\,dr\right) \\ &\le C(n,d)At^{d/2}. \end{aligned}$$

Here we have used the fact that

$$\int_0^\infty r^{n+d}e^{-r^2/((4+\varepsilon)t)}2r\,dr = \Gamma\left(\frac{n+d}{2}+1\right)((4+\varepsilon)t)^{(n+d)/2+1}.$$

The condition $d > -n-2$ is used here to have the integrability of the left hand side above near $r = 0$. To prove the necessary part we use the lower bound estimate of Li-Yau on the heat kernel.

$$\begin{aligned} At^{d/2} \ge u(x,t) &\ge C^{-1}(n)\frac{1}{V_x(\sqrt{t})}\int_0^\infty e^{-r^2/((4+\varepsilon)t)}\int_{\partial B_X(r)} f(y)\,dA_y \\ &\ge C^{-1}(n)\frac{1}{V_x(\sqrt{t})}\int_0^{\sqrt{t}} e^{-r^2/((4+\varepsilon)t)}\int_{\partial B_X(r)} f(y)\,dA_y \\ &\ge C^{-1}(n)\frac{1}{V_x(\sqrt{t})}\int_{B_X(\sqrt{t})} f(y)\,dv_y. \end{aligned}$$

The proof for the logarithmic case needs a little more care on the calculation. As before, by Li-Tau's upper bound on the heat kernel we have that

$$u(x,t) \le C(n)k(x,\sqrt{t}) + \frac{C(n)}{V_x(\sqrt{t})}\int_{\sqrt{t}}^\infty e^{-r^2/(5t)}\int_{\partial B_X(r)} f(y)\,dA.$$

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The first term is controlled by $C(n)A\log(1+t)$ by the assumption that $k(x,r) \le \log(1+r)$. Doing the integration by parts as above, we have the following estimate for the second part.

$$\begin{aligned}
&\frac{C(n)}{V_x(\sqrt{t})}\int_{\sqrt{t}}^{\infty} e^{-r^2/(5t)}\int_{\partial B_x(r)} f(y)\,dA \\
&\quad\le \frac{C(n)}{V_x(\sqrt{t})}\int_{\sqrt{t}}^{\infty}\left(\int_{B_x(r)} f(y)\,dy\right)e^{-r^2/(5t)}\left(\frac{2r}{5t}\right)dr \\
&\quad\le \frac{C(n)A}{t^{n/2+1}}\int_{\sqrt{t}}^{\infty}\log(1+r)r^{n+1}e^{-r^2/(5t)}\,dr \\
&\quad\le \frac{C(n)A}{t^{n/2+1}}\int_{0}^{\infty}\log(1+\tau)\tau^{n/2}e^{-\tau/(5t)}\,d\tau \\
&\quad\le C(n)A\int_{0}^{\infty}\log(1+5ts)s^{n/2}e^{-s}\,ds \\
&\quad\le C(n)A\log(1+t)\int_{0}^{\infty} s^{n/2+1}e^{-s}\,ds \le C(n)A\log(1+t).
\end{aligned}$$

The other direction is simpler and very similar to the previous case. □

Note that, if in the above estimate (of Theorem 3.1) we choose $f(x)$ to be $r(x)$, the distance function to a fixed point $o \in M$, we obtain the moment estimate on complete Riemannian manifolds with nonnegative Ricci curvature. This was first proved by Nash for the Euclidean case in [18]. In [16], Li-Tam proved an upper bound estimate on the solution to the heat equation on general Riemannian manifolds with lower bound on the Ricci curvature. As a corollary we have the following result on the asymptotic behavior of the solution to the heat equation.

Corollary 3.2. *In the case $d = 0$, there exists a constant $C = C(n)$ such that*

$$C^{-1}(n)k(x,\sqrt{t}) \le u(x,t) \le C(n)\sup_{r\ge\sqrt{t}} k(x,r).$$

In particular, $\lim_{t\to\infty} u(x,t) = 0$ *if and only if* $\lim_{r\to\infty} k(x,r) = 0$.

Corollary 3.3. *Let M be a complete Riemannian manifold with nonnegative Ricci curvature. Suppose that $u(x,t)$ is the solution of the heat equation*

$$\begin{cases}\left(\Delta - \dfrac{\partial}{\partial t}\right)u(x,t) = 0,\\ u(x,0) = f(x).\end{cases}$$

Assume that, for some constant a,

$$\lim_{r\to\infty}\frac{1}{V_x(r)}\int_{B_x(r)} |f(y) - a|\,dy = 0. \tag{3.2}$$

Then $\lim_{t\to\infty} u(x,t) = a$.

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Proof. By the assumption we know that both $(f-a)_+ = \max\{0,(f-a)\}$ and $(f-a)_- = -\min\{0,(f-a)\}$ satisfy (2.1). Let $u_+^*(x,t)$ ($u_-^*(x,t)$) be the solution with the initial data $(f-a)_+$ ($(f-a)_-$). By Corollary 3.2 we have that

$$0 \le u_+^*(x,t) \le C(n) \sup_{r\ge\sqrt{t}} \left(\frac{1}{V_x(r)} \int_{B_x(r)} (f-a)_+(y)\, dy \right) \to 0, \quad \text{as } t \to \infty.$$

Therefore $\lim_{t\to\infty} u_+^*(x,t) = 0$. Similarly we have $\lim_{t\to\infty} u_-^*(x,t) = 0$. This also holds for $u^*(x,t) = u_+^* + u_-^*$, which solves the heat equation with the initial data $f-a$. Our statement is an easy consequence of it. □

This partially generalizes Li's theorem (cf. [14, Theorem 3]) since we need not assume either that M is of maximum volume growth or that f is bounded. But our assumption (3.1) on the initial data is stronger than the assumption in Theorem 3 of [14].

In order to study the Hermitian-Einstein heat equation we need the following result, which is the parabolic version of Lemma 2.4.

Proposition 3.4. *Let M be a complete Riemannian manifold with nonnegative Ricci curvature. Let f be a nonnegative continuous function on M. Consider the heat equation*

$$\begin{cases} \left(\Delta - \dfrac{\partial}{\partial t}\right) v(x,t) = -f(x), \\ v(x,0) = 0. \end{cases}$$

Then it has a nonnegative solution $v(x,t)$ such that $\sup_{t\ge0} v(x,t) < \infty$ if and only if

$$\int_0^\infty \frac{r}{V_x(r)} \int_{B_x(r)} f(y)\, dy\, dr < \infty.$$

Proof. The case when M is parabolic is the same as in Lemma 2.4. We just focus on the nonparabolic case. We start with the necessary part. Suppose that there is a solution $v(x,t)$ such that $\sup_{t\ge0} v(x,t) < \infty$. $v_t(x,t)$ is a nonnegative solution of the heat equation such that $\int_0^\infty (v_t)\, dt < \infty$ by the assumption. Therefore there will be a sequence t_i such that $v(x,t_i)$ converges over a compact subset of M. By passing to a subsequence we then obtain a nonnegative solution of the Poisson equation by taking $v_\infty(x) = \lim_{i\to\infty} v(x,t_i)$. Applying Lemma 2.4 we prove the necessary part. To prove the sufficient part we do the following estimates.

$$\begin{aligned} v(x,t) &\le \int_0^t \int_M H(x,y,\tau) f(y)\, dv_y\, d\tau \\ &\le C(n) \int_0^\infty \int_0^\infty \frac{1}{V_x(\sqrt{t})} e^{-r^2/((4+\varepsilon)t)} \int_{\partial B_x(r)} f(y)\, dA_y\, dr\, dt. \end{aligned}$$

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Here we have used the heat kernel estimate of Li-Yau. Using Fubini theorem we have that

$$\begin{aligned} v(x,t) &\le C(n)\int_0^\infty \left(\int_0^\infty \frac{1}{V_x(\sqrt{t})} e^{-r^2/((4+\varepsilon)t)}\,dt\right)\left(\int_{\partial B_x(r)} f(y)\,dA\right)dr \\ &= C(n)\int_0^\infty \left(\int_0^{r^2} \frac{1}{V_x(\sqrt{t})} e^{-r^2/((4+\varepsilon)t)}\,dt\right)\left(\int_{\partial B_x(r)} f(y)\,dA\right)dr \\ &\quad + C(n)\int_0^\infty \left(\int_{r^2}^\infty \frac{1}{V_x(\sqrt{t})} e^{-r^2/((4+\varepsilon)t)}\,dt\right)\left(\int_{\partial B_x(r)} f(y)\,dA\right)dr \\ &= I + II. \end{aligned}$$

Applying the estimate in the proof of Theorem 5.2 of [17], we have that

$$\begin{aligned} I + II &\le C(n)\int_0^\infty \left(\int_{r^2}^\infty \frac{1}{V_x(\sqrt{t})}\,dt\right)\left(\int_{\partial B_x(r)} f(y)\,dA\right)dr \\ &= C(n)\int_0^\infty \left(\int_r^\infty \frac{t}{V_x(t)}\,dt\right)\left(\int_{\partial B_x(r)} f(y)\,dA\right)dr. \end{aligned}$$

Applying Fubini's theorem one more time as in the proof of Lemma 2.4, we have

$$v(x,t) \le C(n)\int_0^\infty \frac{r}{V_x(r)}\int_{B_X(r)} f(y)\,dv_y\,dr < \infty. \qquad \square$$

Letting $u(x,t) = v_t(x,t)$ we can write the following corollary.

Corollary 3.5. *Let M be a complete Riemannian manifold with nonnegative Ricci curvature. Let f be a nonnegative continuous function on M. Consider the heat equation*

$$\begin{cases} \left(\Delta - \dfrac{\partial}{\partial t}\right) u(x,t) = 0, \\ u(x,0) = f(x). \end{cases}$$

Then it has a nonnegative solution $u(x,t)$ such that $\int_0^\infty u(x,s)\,ds < \infty$ if and only if

$$\int_0^\infty \frac{r}{V_x(r)}\int_{B_X(r)} f(y)\,dy\,dr < \infty.$$

We should point out that, if we replace the non-negativity of the Ricci curvature by the assumption that M satisfies the L^2-Sobolev inequality, the similar result holds. In fact we have the following result.

Theorem 3.6. *Let M be a complete Riemannian manifold of dimension n, which supports the L^2-Sobolev inequality. Let f be a nonnegative continuous function on M. Denote*

$$\tilde{k}(x,r) = \frac{1}{r^n}\int_{B_X(r)} f(y)\,dy.$$

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Consider the solution $u(x,t)$ *to the scalar heat equation with the initial data* $u(x,0) = f(x)$. *We have that*

(i) *if* $\tilde{k}(x,r) \le Ar^{2d}$ *for some* $-n/2 < d < \infty$, *then there exists a constant* $C(n,d)$ *such that* $u(x,t) \le C(n,d)At^{d/2}$;

(ii) *if* $\lim_{r\to\infty} \tilde{k}(x,r) = 0$, *then* $\lim_{t\to\infty} u(x,t) = 0$.

The proof of the above result is very similar to that of Theorem 3.1. When $f(x) = r(o,x)$, the distance function to a point $o \in M$, Nash's moment estimate in [18] is a two-sides estimate, which is similar to Theorem 3.1 and better than the above result. Even his argument is applicable to general manifolds, it does not seem to work if one replace $r(o,x)$ by a general function $f(x)$. In this sense, our result is more general.

4. Hermitian-Einstein heat equation

In this section we are going to prove a long time existence for the Hermitian-Einstein flow under a very relaxed condition on the initial metric on any complete Kähler manifold. Using the results from last section we shall also give a heat flow proof of Theorem 2.10.

Theorem 4.1. *Let* M *be a complete Kähler manifold, and let* E *be a holomorphic vector bundle of* $\operatorname{rank} k$ *over* M *with metric* H_0. *If there is a constant* λ *such that the mean curvature* ΛF_{H_0} *satisfies* $\|\Lambda F_{H_0} - \lambda I\| \le A$ *for some positive number* A, *then the Hermitian-Einstein flow*

$$\begin{cases} \dfrac{\partial h}{\partial t}h^{-1} = -\Lambda F_H + \lambda I, \\ h(0) = I \end{cases}$$

has a long time solution on $M \times [0,\infty)$. *Here* $h(x,t) = H(x,t)H_0^{-1}(x)$. *Note that we do not assume that* M *has nonnegative Ricci curvature.*

As before, we use the compact exhaustion construction to prove the long time existence. Also, we will only give the proof for the case $\lambda = 0$. Let Ω_i be a compact exhaustion of M. By Donaldson's theorem we know that the following Dirichlet problem is solvable:

$$(\mathrm{D}_i) \qquad \begin{cases} \dfrac{\partial h}{\partial t}h^{-1} = -\Lambda F_H, \\ H(x,0) = H_0(x), \\ H(x,t)\big|_{\partial\Omega_i} = H_0(x). \end{cases}$$

Denote by $H_i(x,t)$ the solution to the above Dirichlet problem. To prove the theorem we need to show that there exists a subsequence of H_i that converges to a solution defined on the whole manifold M.

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Before we prove the theorem we need a couple of lemmas. The following lemmas were proved in [9].

Let $v(x,t) = \log\det(HH_0^{-1})(x,t)$, $R(x,t) = \operatorname{tr}\Lambda F_H$, and $R(x) = \operatorname{tr}\Lambda F_{H_0}$. The first lemma is on the equation of ΛF_H

Lemma 4.2.

$$\left(\Delta - \frac{\partial}{\partial t}\right)\Lambda F_H = 0, \quad \text{and} \tag{4.1}$$

$$\left(\Delta - \frac{\partial}{\partial t}\right)R(x,t) = 0. \tag{4.2}$$

An easy consequence of above lemma is the following result.

Lemma 4.3. *Let $H(t)$ be a solution to the Hermitian-Einstein flow. Then $\hat{e}(x,t) = \|\Lambda F_{H(t)}\|$ satisfies*

$$\Delta\hat{e} - \frac{\partial\hat{e}}{\partial t} \geq 0.$$

From the maximum principle we have that $\hat{e}_i(x,t) = \|\Lambda F_{H_i(t)}\|$, where $H_i(x,t)$ are solutions to (D_i), satisfies

$$\sup_{\Omega_i\times[0,\infty)} \hat{e}_i \leq A. \tag{4.3}$$

Another consequence of Lemma 4.2 is the following corollary.

Corollary 4.4.

$$\left(\Delta - \frac{\partial}{\partial t}\right)v(x,t) = R(x). \tag{4.4}$$

Proof. First, from the definition of the so-called Bott-Chern class [3], it is easy to see that

$$\operatorname{tr}F_H - \operatorname{tr}F_{H_0} = \bar{\partial}\partial\int_0^t \operatorname{tr}\left(\frac{\partial h}{\partial t}h^{-1}\right)(x,s)\,ds.$$

Using the fact

$$\frac{\partial}{\partial t}\log\det(h) = \operatorname{tr}\left(\frac{\partial h}{\partial t}h^{-1}\right),$$

we can conclude that

$$\bar{\partial}\partial\log\det(h) = \operatorname{tr}F_H - \operatorname{tr}F_{H_0}. \tag{4.5}$$

Now it is easy to see that

$$\Delta v(x,t) = R(x) - R(x,t). \tag{4.6}$$

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Direct calculation also shows that

$$v_t = \mathrm{tr}\left(\frac{\partial h}{\partial t}h^{-1}\right) = -R(x,t),$$

from which our claim follows. □

Let H and K to be two Hermitian metrics on E. Then HK^{-1} is a section of $\mathrm{Hom}(E,E)$. Recall that Donaldson's "distance" functions $\tau(H,K)$ and $\sigma(H,K)$ are defined as $\tau(H,K) = \mathrm{tr}(HK^{-1})$ and $\sigma(H,K) = \tau(H,K)+\tau(K,H)-2\,\mathrm{rank}(E)$. The function $\tau(H(x,t),H_0(x))$ satisfies the following lemma.

Lemma 4.5. *Let $H(x,t)$ be a solution to Hermitian-Einstein flow with initial metric $H_0(x)$. Then*

$$\left(\Delta - \frac{\partial}{\partial t}\right)\log\tau(H,H_0) \geq -\|\Lambda F_{H_0}\|.$$

As a corollary, if we denote $\rho_i(x,t) = \log\tau(H_i,H_0) - \log(\mathrm{rank}(E))$, then we have the following

$$\begin{cases} \left(\Delta - \dfrac{\partial}{\partial t}\right)\rho_i(x,t) \geq -\hat{e}_0(x), \\ \rho_i(x,0) = 0, \\ \rho_i(x,t)\big|_{\partial\Omega_i} = 0. \end{cases} \tag{4.7}$$

Here $\hat{e}_0(x) = \|\Lambda F_{H_0}\|$. Now applying the maximum principle to $e^{-t}\rho_i$, noticing that $(\Delta - \partial/\partial t)\rho_i \geq -A$, we can have the following

$$\sup_{\Omega_i\times[0,T]} \rho_i(x,t) \leq e^T A. \tag{4.8}$$

Proof of Theorem 4.1. Denote $v_i(x,t) = \log\det(H_iH_0^{-1})$. Direct calculation shows that

$$\frac{\partial v_i}{\partial t} = \mathrm{tr}(-\Lambda F_{H_i}).$$

Together with Lemma 4.3, this implies that

$$\frac{\partial v_i}{\partial t} \geq -A.$$

Therefore $\det(H_iH_0^{-1})(x,t) \geq e^{-At}$. Using the above inequality and (4.8), we have that

$$\sup_{\Omega_i\times[0,T]} \mathrm{tr}(H_0H_i^{-1}) \leq C_1(T,A).$$

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This would imply that

$$\sup_{\Omega_i \times [0,T]} \sigma(H_i, H_0) \le C_2(T, A). \tag{4.9}$$

In particular, over any compact subset K, we have the so-called C^0-estimate

$$\sup_{K \times [0,T]} \sigma(H_i, H_0) \le C_2(T, A). \tag{4.10}$$

Once we have established the C^0-estimates, we can adapt the argument in [20] to conclude that there exists a subsequence of $H_i(x,t)$ such that it converges to a solution to the Hermitian-Einstein flow (cf. [20, Theorem 3.8]). The argument is basically the following. From (4.3), together with a similar differential inequality on τ_i, one can establish the integral bound for the gradient. Then the Sobolev embedding implies the L^2-convergence, then the C^0-convergence of H_i, after applying the mean value inequality of Li-Tam [16] for the nonnegative subsolutions to the heat equation. Then one can use Donaldson's scaling argument to obtain the C^0-bounds for the gradients. The argument in the last part of the proof of Theorem 2.10 also works here. This finishes the proof of Theorem 4.1. □

Theorem 4.6. *Let M be a complete Kähler manifold with nonnegative Ricci curvature, and let E be a holomorphic vector bundle of* rank k *over M with metric H_0 such that $\|\Lambda F_{H_0}\|$ satisfies, for every $x \in M$,*

$$\int_0^\infty \frac{t}{V_x(t)} \int_{B_x(t)} \|\Lambda F_{H_0}\|(y)\, dy < \infty.$$

Then the Hermitian-Einstein flow

$$\begin{cases} \dfrac{\partial h}{\partial t} h^{-1} = -\Lambda F_H, \\ h(0) = I. \end{cases}$$

has a long time solution on $M \times [0, \infty)$, and $H(x,t)$ converges to a solution to the Hermitian-Einstein equation.

Proof. The key point is to derive an upper bound estimate on $\sigma(H(x,t), H_0(x))$ which is independent of t. It is at this point we need to use some sharper estimates than the one used in the proof of Theorem 4.1. Let $v^*(x,t)$ be the solution to the equation

$$\begin{cases} \left(\Delta - \dfrac{\partial}{\partial t}\right) v^*(x,t) = -\hat{e}_0(x), \\ v^*(x,t) = 0. \end{cases}$$

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By Proposition 3.4 we know that $\sup_{t\geq 0} v^*(x,t) < \infty$. From Lemma 4.5 and the comparison principle we know that $\tau(H(x,t),H_0(x)) \leq ke^{v^*(x,t)}$. We give another derivation of the estimate for $\sigma(H(x,t),H_0(x))$, which makes use of the function $v(x,t) = \log\det(HH_0^{-1})(x,t)$. Using the fact $v(x,0) = 0$ and the comparison principle, we have that $-v(x,t) \leq v^*(x,t)$. This gives a lower bound on $\det(HH_0^{-1})(x,t)$. Together with the estimate on $\tau(H(x,t),H_0(x))$ we have a upper bound estimate for $\tau(H_0(X),H(x,t))$ by simple algebraic consideration, and then the upper bound estimate of $\sigma(H(x,t),H_0(x))$. If we use the fact that $\tau(H_0(x),H(x,t))$ also satisfies Lemma 4.5 we can, as in the elliptic case, have a more precise estimate

$$\sigma(H(x,t),H_0(x)) \leq 2k(e^{v^*(x,t)} - 1).$$

Applying Proposition 3.4 we have that on any compact subset $K \subset M$ there is a uniform upper bound for $\sigma(H(x,t),H_0(x))$. Together with the gradient estimate we derived in the proof of Theorem 2.10 we have at least t_i such that $H(x,t_i)$ converges to a metric $H_\infty(x)$. Using Lemma 4.3 and the fact $\hat{e}(x,0) = \|\Lambda F_{H_0}\|$, by Corollary 3.3 and the comparison principle we have that $\lim_{t\to\infty}\hat{e}(x,t) = 0$. Therefore $H_\infty(x)$ is a solution to the Hermitian-Einstein equation. The convergence of $H(x,t)$ to $H_\infty(x)$ follows from a more careful estimate of $\sigma(H(x,t),H_0(x))$ as shown below.

Let $t_1 \leq t \leq t_2$ and let $\hat{h}(x,t) = H(x,t)H^{-1}(x,t_1)$. It is an easy matter of checking to see that

$$\frac{\partial\hat{h}}{\partial t}\hat{h}^{-1} = -\Lambda F_H.$$

From here it is easy to see that

$$\begin{aligned}\frac{\partial}{\partial t}\log\mathrm{tr}(\hat{h}) &= \frac{\mathrm{tr}\left(H_t(x,t)H^{-1}(x,t_1)\right)}{\mathrm{tr}(H(x,t)H^{-1}(x,t_1))}\\ &= \frac{\mathrm{tr}(-\Lambda F_H H(x,t)H^{-1}(x,t_1))}{\mathrm{tr}\left(H(x,t)H^{-1}(x,t_1)\right)} \leq |\Lambda F_H|.\end{aligned}$$

Then we have that

$$\tau(H(x,t),H(x,t_1)) \leq ke^{\int_{t_1}^t |\Lambda F_H|(x,s)\,ds}. \tag{4.11}$$

We have a similar estimate for $\tau(H(x,t_1),H(x,t))$. Combining them we have

$$\sigma(H(x,t),H(x,t_1)) \leq 2k(e^{\int_{t_1}^t |\Lambda F_H|(x,s)\,ds} - 1).$$

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Let $u(x,t)$ be a solution of the heat equation with $u(x,0) = \hat{e}_0(x)$. Then by Corollary 3.5 and Lemma 4.2 we know that

$$\begin{aligned}\sigma(H(x,t_2),H(x,t_1)) &\le 2k(e^{\int_{t_1}^{t_2} u(x,s)\,ds} - 1)\\ &\le 2k(e^{\int_{t_1}^{\infty} u(x,s)\,ds} - 1) \to 0, \quad \text{as } t_1 \to \infty.\end{aligned}$$

Thus we have proved that $H(x,t)$ converges to $H_\infty(x)$. □

Remark. When M has $\lambda_1(M) > 0$, the heat flow proof of the Hermitian-Einstein metrics was given in [20]. If M satisfies the L^2-Sobolev inequality, one can provide a heat flow proof of the existence of the Hermitian-Einstein metrics if the mean curvature of the initial metric has certain decay conditions in terms of $\tilde{k}(x,r)$, which is defined in Theorem 3.6 of the last section. The result will be slightly different from the stated result in Theorem 2.9. We leave the details to the interested readers.

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