Sharp Differential Estimates of Li-Yau-Hamilton Type for Positive (p, p)-Forms on Kähler Manifolds

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Abstract

In this paper we study the heat equation (of Hodge Laplacian) deformation of (p, p)-forms on a Kähler manifold. After identifying the condition and establishing that the positivity of a (p, p)-form solution is preserved under such an invariant condition, we prove the sharp differential Harnack (in the sense of Li-Yau-Hamilton) estimates for the positive solutions of the Hodge Laplacian heat equation. We also prove a nonlinear version coupled with the Kähler-Ricci flow and some interpolating matrix differential Harnack-type estimates for both the Kähler-Ricci flow and the Ricci flow. © 2011 Wiley Periodicals, Inc.

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1 Introduction

In this paper, we study the deformation of positive (p, p)-forms on a Kähler manifold via the $\overline{\partial}$-Laplacian heat equation. One of our main goals is to prove

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differential Harnack estimates for positive solutions. The Harnack estimate for positive solutions of linear parabolic PDEs of divergence form goes back to the fundamental work of Moser [28]. In another fundamental work [23], Li and Yau, on Riemannian manifolds with nonnegative Ricci curvature, proved a sharp differential estimate that implies a sharp form of Harnack inequality for the positive solutions.

Later, Hamilton [17] proved the miraculous matrix differential estimates for the curvature operator of solutions to Ricci flow when the curvature operator is nonnegative. Since the curvature operator of a Ricci flow solution satisfies a nonlinear diffusion-reaction equation, this result of Hamilton is as surprising as it is important. Due to this development people also call this type of sharp estimate "Li-Yau-Hamilton type" (abbreviated as LYH type) estimates. There are many additional works [2, 5, 8, 9, 10, 11, 19, 18, 30, 32, 34] in this direction since the foundational estimate of Li and Yau for linear heat equation and Hamilton's one for the Ricci flow, which cover various different geometric evolution equations, including the mean curvature flow, the Gauss curvature flow, the Kähler-Ricci flow, the Hermitian-Einstein flow, the Yamabe flow, etc.

Since the Harnack estimate for the linear equation implies the regularity of the weak solution, it has been an interesting question whether the celebrated De Giorgi–Nash–Moser theory for the linear equation has an analogue for linear systems. This unfortunately has been known to be false in the most general setting. As a geometrically interesting system, the Hodge Laplacian operator on forms has been extensively studied since the original works of Hodge and Kodaira (see, for example, Morrey's classics [26] and references therein). It is a natural candidate on which one would like to investigate whether the differential Harnack estimates of LYH type still hold. One of the main results of this paper is to prove such LYH type estimates for this system.

The positivity (really meaning nonnegativity) of the (p, p)-form is in the sense of Lelong [21]. In fact, in [30] the first author proved a LYH type estimate for positive semidefinite Hermitian symmetric tensors satisfying the so-called Lichnerowicz Laplacian heat equation. This in particular applies to solutions of $(1, 1)$-forms to the Hodge Laplacian heat equation. The first main result of this paper is to generalize this result for $(1, 1)$-forms to solutions of (p, p)-forms to the Hodge Laplacian heat equation. Such a generalization is also motivated by the Bishop-Lelong monotonicity for positive (p, p)-currents on complex Euclidean spaces (see, for example, [33]). The result is proved under a new curvature condition $\mathcal{C}_p$. We say that the curvature operator Rm of a Kähler manifold (M, g) satisfies $\mathcal{C}_p$ (or lies inside the cone $\mathcal{C}_p$) if $\langle \mathrm{Rm}(\alpha), \overline{\alpha} \rangle \geq 0$ for any $\alpha \in \bigwedge^{1,1}(\mathbb{C}^m)$ such that $\alpha = \sum_{k=1}^{p} X_k \wedge \overline{Y}_k$ with $X_k, Y_k \in T'M$. Here $TM \otimes \mathbb{C} = T'M \oplus T''M$ and $\langle \cdot, \cdot \rangle$ is the bilinear extension of the Riemannian product, and we identify $T'M$ with $\mathbb{C}^m$. Under the condition $\mathcal{C}_p$ we prove the following result:

THEOREM 1.1 *For $\phi(x,t)$, defined as a positive (p,p)-form satisfying $(\frac{\partial}{\partial t} + \Delta_{\overline{\partial}})\phi(x,t) = 0$,*

$$\frac{1}{\sqrt{-1}}\,\overline{\partial}^*\partial^*\phi + \frac{1}{\sqrt{-1}}\,\iota_V\cdot\overline{\partial}^*\phi - \frac{1}{\sqrt{-1}}\,\iota_{\overline{V}}\cdot\partial^*\phi + \sqrt{-1}\iota_V\cdot\iota_{\overline{V}}\cdot\phi + \frac{\Lambda\phi}{t} \geq 0$$

holds as a $(p-1,p-1)$-form, for any $(1,0)$-type vector field V. Here Λ is the adjoint of the operator $L \doteqdot \omega\wedge(\cdot)$ with ω being the Kähler form.

The above estimate is compatible with the Hodge $*$ operator (see Section 4 for a detailed discussion). Also note the easy fact that the $*$-operator maps a positive (p,p)-form to a positive $(m-p,m-p)$-form. The above result then implies that if $\psi = *\phi$,

$$\sqrt{-1}\partial\overline{\partial}\psi + \sqrt{-1}V^*\wedge\overline{\partial}\psi - \sqrt{-1}\,\overline{V}^*\wedge\partial\psi + \sqrt{-1}V^*\wedge\overline{V}^*\wedge\psi + \frac{L(\psi)}{t} \geq 0$$

as a $(m-p+1,m-p+1)$-form, with V^* being a $(1,0)$-type 1-form. This generalizes the matrix estimate for positive solutions to the heat equation proved in [9], which yields $\sqrt{-1}\,\partial\overline{\partial}\log\psi + \omega/t \geq 0$. (Note that this is the matrix version of Li and Yau's estimate, $\Delta\log\psi + m/t \geq 0$. See also [42] for earlier work on harmonic functions.) The proof is a combination of techniques of [8, 30, 35] and Hamilton's argument in [17]. Applying the above result to the static solution yields a differential estimate:

$$\frac{1}{\sqrt{-1}}\,\overline{\partial}^*\partial^*\phi + \frac{1}{\sqrt{-1}}\,\iota_V\cdot\overline{\partial}^*\phi - \frac{1}{\sqrt{-1}}\,\iota_{\overline{V}}\cdot\partial^*\phi + \sqrt{-1}\,\iota_V\cdot\iota_{\overline{V}}\cdot\phi \geq 0 \tag{1.1}$$

for any $\overline{\partial}$-harmonic positive (p,p)-form ϕ and any vector field V of $(1,0)$-type. Note here on a noncompact manifold, being harmonic does not imply that $\partial^*\phi = 0$ (or $\overline{\partial}^*\phi = 0$).

As a result of independent interest, we also observe that $\mathcal{C}_p$ is an invariant condition under the Kähler-Ricci flow, thanks to a general invariant cone result of Wilking [41], whose proof we include here in the Appendix (see also a recent preprint [7]). Note that this result of Wilking includes almost all the known invariant cones such as the nonnegativity of bisectional curvature [3, 24], the nonnegativity of isotropic curvature [6, 29], etc.

After establishing the invariance of $\mathcal{C}_p$, it is natural to study the heat equation for the Hodge Laplacian coupled with the Kähler-Ricci flow. For this we proved the following nonlinear version of the above estimate:

THEOREM 1.2 *Assume that $\phi(x,t) \geq 0$ is a solution to the heat equation of the Hodge Laplacian coupled with the Kähler-Ricci flow: $\frac{\partial}{\partial t}g_{i\overline{j}} = -R_{i\overline{j}}$. Then*

$$\frac{1}{\sqrt{-1}}(\overline{\partial}^*\partial^*\phi + \iota_V\cdot\overline{\partial}^*\phi - \iota_{\overline{V}}\cdot\partial^*\phi - \iota_V\cdot\iota_{\overline{V}}\cdot\phi) + \Lambda_{\mathrm{Ric}}(\phi) + \frac{\Lambda\phi}{t} \geq 0$$

as a $(p-1,p-1)$-form for any $(1,0)$-type vector field V. Here $\Lambda_{\mathrm{Ric}}\phi$ is the adjoint of $\mathrm{Ric}\wedge(\cdot)$ *with* $\mathrm{Ric} = \sqrt{-1}\,R_{i\overline{j}}\,dz^i\wedge dz^{\overline{j}}$ *being the Ricci form.*

To prove the above result, it is necessary to prove the following family of matrix differential estimates that interpolate between Hamilton's matrix estimate in [17] and Cao's estimate for the Kähler-Ricci flow in [8].

THEOREM 1.3 *Let $(M, g(t))$ be a complete solution to the Kähler-Ricci flow satisfying the condition $\mathcal{C}_p$ on $M \times [0, T]$. When M is noncompact, we assume that the curvature of $(M, g(t))$ is bounded on $M \times [0, T]$. Then for any $\bigwedge^{1,1}$-vector U that can be written as $U = \sum_{i=1}^{p-1} X_i \wedge \overline{Y}_i + W \wedge \overline{V}$, for $(1,0)$-type vectors X_i, Y_i, W, and V, the Hermitian bilinear form*

$$\mathcal{Q}(U \oplus W) \doteqdot \mathcal{M}_{\alpha\bar{\beta}} W^{\alpha} W^{\bar{\beta}} + P_{\alpha\bar{\beta}\gamma} \bar{U}^{\bar{\beta}\gamma} W^{\alpha} + P_{\alpha\bar{\beta}\bar{\gamma}} U^{\alpha\bar{\gamma}} W^{\bar{\beta}} + R_{\alpha\bar{\beta}\gamma\bar{\delta}} U^{\alpha\bar{\beta}} \bar{U}^{\bar{\delta}\gamma}$$

satisfies that $\mathcal{Q} \geq 0$ for any $t > 0$. Moreover, if the equality ever occurs for some $t > 0$, the universal cover of $(M, g(t))$ must be a gradient-expanding Kähler-Ricci soliton.

Recall that the tensors $\mathcal{M}$ and P are defined as

$$\mathcal{M}_{\alpha\bar{\beta}} = \Delta R_{\alpha\bar{\beta}} + R_{\alpha\bar{\beta}\gamma\bar{\delta}} R_{\delta\bar{\gamma}} + \frac{R_{\alpha\bar{\beta}}}{t}, \quad P_{\alpha\bar{\beta}\gamma} = \nabla_{\gamma} R_{\alpha\bar{\beta}}, \quad P_{\alpha\bar{\beta}\bar{\gamma}} = \nabla_{\bar{\gamma}} R_{\alpha\bar{\beta}}.$$

There exists a similar condition $\widetilde{\mathcal{C}}_p$ for a Riemannian manifold, which can be formulated similarly. To be precise, we say that the curvature operator satisfies $\widetilde{\mathcal{C}}_p$ if $\langle \mathrm{Rm}(v), \overline{v} \rangle > 0$ for any nonzero $v \in \Lambda^2(\mathbb{C}^n)$ that can be written as $v = \sum_{i=1}^{p} Z_i \wedge W_i$ for some complex vectors Z_i and $W_i \in TM \otimes \mathbb{C}$. For Kähler manifolds it can be shown that $\widetilde{\mathcal{C}}_p = \mathcal{C}_{2p}$ and being inside $\mathcal{C}_2$ implies the nonnegativity of the complex sectional curvature, a notion that goes back at least to the work of Sampson [37] on harmonic maps. This leads us to discover another family of matrix differential estimates for the Ricci flow that interpolate the result of Hamilton [17] and a recent result of Brendle [5].

THEOREM 1.4 *Assume that $(M, g(t))$ on $M \times [0, T]$ satisfies $\widetilde{\mathcal{C}}_p$. When M is noncompact we also assume that the curvature of $(M, g(t))$ is uniformly bounded on $M \times [0, T]$. Then for any $t > 0$, the quadratic form*

$$\widetilde{\mathcal{Q}}(W \oplus U) \doteqdot \langle \mathcal{M}(W), \overline{W} \rangle + 2\,\mathrm{Re}(\langle P(W), \overline{U} \rangle) + \langle \mathrm{Rm}(U), \overline{U} \rangle$$

satisfies that $\widetilde{\mathcal{Q}} \geq 0$ for any $(x, t) \in M \times [0, T]$, $W \in T_x M \otimes \mathbb{C}$, and $U \in \bigwedge^2 (T_x M \otimes \mathbb{C})$ such that $U = \sum_{\mu=1}^{p} W_\mu \wedge Z_\mu$ with $W_p = W$. Furthermore, that the equality holds for some $t > 0$ implies that the universal cover of $(M, g(t))$ is a gradient-expanding Ricci soliton.

Here $\mathcal{M}$ and P are defined similarly. In fact, for $p = 1$, our result is slightly stronger than Brendle's estimate [5]. After we finished our paper, a recent preprint [7] was brought to our attention, where a similar, but seemingly more general, result was formulated in terms of the space-time consideration of Chow and Chu [12].

Nevertheless, our proof here is direct and explicit without involving the space-time formulation. The proof is in fact rather short (see Section 9) and is motivated by the Kähler case.

Compared with the previous work on $(1, 1)$-forms in [30, 34], in this paper we do not assume any growth conditions at infinity for the LYH estimate on the *positive* (p, p)-forms. Hence even for the $p = 1$ case, Theorem 1.1 here is an improvement on the earlier works. The main reason for such an improvement is the observation that the *positivity* (which is assumed when we prove the LYH estimate) itself implies a growth condition as in the classical Widder's proof of the uniqueness of the positive solution to the (scalar) heat equation. Here the growth condition for a positive solution to the Hodge Laplacian heat equation is obtained via Li and Yau's fundamental estimates on the positive solutions to the scalar heat equation on a Riemannian manifold with nonnegative Ricci curvature.

On the other hand, when one solves the Cauchy problem for the Hodge Laplacian heat equation, in order to ensure the global existence, one does need some growth assumption for the initial data. Similarly, due to the nonuniqueness of the heat equation in general, in order to prove that the positivity is *preserved* under the Hodge Laplacian heat equation (assuming only the positivity of the initial data), one has to impose a condition similar to the scalar heat equation case since without such an assumption even for zero data there are nonzero solutions taking both positive and negative values. These issues are discussed in detail in Section 11. Comparison with the heat equation on Euclidean spaces shows that the growth conditions imposed for a global solution to the Cauchy problem in Section 11, as well as for the preservation of the positivity of a (p, p)-form solution to the Hodge Laplacian heat equation, are optimal ones. This part extends the earlier work of [31] and [35]. Curiously enough, due to the lack of an estimate for the scalar curvature without the matrix estimate, we do not know if similar growth assumptions can be removed for Hamilton's matrix LYH estimate for the solutions to the Ricci flow.

Here is how we organize the paper. In Section 2 we prove that under the condition $\mathcal{C}_p$ the positivity of the (p, p)-forms is preserved under the Hodge Laplacian heat equation. In Section 3 we derive the invariance of $\mathcal{C}_p$ by refining an argument of Wilking, which is detailed in the Appendix. In Section 4 we collect and prove some preliminary formulae needed for the proof of the Theorem 1.1. The rigidity on the equality case as well as a monotonicity formula implied by Theorem 1.1 are also included in Section 4. Section 5 is devoted to the proof of Theorem 1.1. Sections 6 and 8 are devoted to the proof of Theorem 1.2 as Section 7 is to the proof of Theorem 1.3, which is needed in Section 8. Section 9 is on the proof of Theorem 1.4. Since up to Section 9 we present only the argument for compact manifolds, Section 10 is the noncompact version of Sections 3, 7, and 9, where the metric is assumed to have bounded curvature, while Section 11 supplies the argument for the noncompact version of Sections 2 and 6, where no upper bound on the curvature is assumed.

Due to the length of the paper we shall study the applications of the estimates in a forthcoming article. We also remark that there exists an interpolation between the estimates of Theorems 1.1 and 1.2 as in [30, 32].

2 Heat Equation Deformation of (p, p)-forms

Let (M^m, g) be a complex Hermitian manifold of complex dimension m. Recall that a (p, p)-form ϕ is called positive if for any $x \in M$ and for any vectors $v_1, \ldots, v_p \in T_x^{1,0}M$, $\langle \phi, \frac{1}{\sqrt{-1}} v_1 \wedge \overline{v}_1 \wedge \cdots \wedge \frac{1}{\sqrt{-1}} v_p \wedge \overline{v}_p \rangle \geq 0$. By linear algebra (see also [39]) it suffices to check that the nonnegativity holds for $v_1, \ldots, v_p$ satisfying that $\langle v_i, v_j \rangle = \delta_{ij}$. We also denote $\langle \phi, \frac{1}{\sqrt{-1}} v_1 \wedge \overline{v}_1 \wedge \cdots \wedge \frac{1}{\sqrt{-1}} v_p \wedge \overline{v}_p \rangle$ by $\phi(v_1, \ldots, v_p; \overline{v}_1, \ldots, \overline{v}_p)$, or even $\phi_{v_1 \cdots v_p, \overline{v}_1 \cdots \overline{v}_p}$. We say that ϕ is strictly positive if $\phi_{v_1 \cdots v_p, \overline{v}_1 \cdots \overline{v}_p}$ is positive for any linearly independent $\{v_i\}_{i=1}^p$.

Let $\Delta_{\overline{\partial}} = \overline{\partial}\overline{\partial}^* + \overline{\partial}^*\overline{\partial}$ be the $\overline{\partial}$-Hodge Laplacian operator. There also exists a Laplacian operator Δ defined by

$$\Delta = \frac{1}{2}(\nabla_i \nabla_{\overline{i}} + \nabla_{\overline{i}} \nabla_i).$$

where ∇ is the induced covariant derivative on (p, p)-forms. Since the complex geometry, analysis, and Riemannian geometry fit better when the manifold is Kähler, we assume that (M, g) is a Kähler manifold for our discussion. Let $\omega = \sqrt{-1}\, g_{i\overline{j}}\, dz^i \wedge dz^{\overline{j}}$ be the Kähler form. Clearly ω^p is a strictly positive (p, p)-form.

For a (p, p)-form ϕ_0, consider the evolution equation

$$\left(\frac{\partial}{\partial t} + \Delta_{\overline{\partial}} \right) \phi(x, t) = 0 \tag{2.1}$$

with initial value $\phi(x, 0) = \phi_0(x)$. Our first concern is when the positivity of the (p, p)-forms is preserved under the above evolution equation. If we denote by $\mathcal{P}_p$ the closed cone consisting of all positive (p, p)-forms, an equivalent question is whether $\mathcal{P}_p$ is preserved under the heat equation (2.1). The answer is well-known for the cases $p = 0$ and $p = m$, since the equation is nothing but the regular heat equation. For $p = 1$, this question was studied in [34] as well as [35], and it was proved that when (M, g) is a complete Kähler manifold with nonnegative bisectional curvature, then the positivity is preserved for the solutions satisfying certain reasonable growth conditions, which is needed for the uniqueness of the solution with the given initial data.

It turns out, to prove the invariance of $\mathcal{P}_p$ for $m - 1 \geq p \geq 2$, we need to introduce a new curvature condition, which we shall formulate below. We say that the curvature operator Rm of a Kähler manifold (M, g) satisfies $\mathcal{C}_p$ (or lies inside the cone $\mathcal{C}_p$) if

$$\langle \mathrm{Rm}(\alpha), \overline{\alpha} \rangle \geq 0$$

for any $\alpha \in \bigwedge^{1,1}(\mathbb{C}^m)$ (we use $\bigwedge^{1,1}_{\mathbb{R}}(\mathbb{C}^m)$ to denote the space of real wedge-2 vectors of $(1,1)$-type), such that it can be written as $\alpha = \sum_{k=1}^{p} X_k \wedge \overline{Y}_k$. Here $TM \otimes \mathbb{C} = T'M \oplus T''M$ and $\langle \cdot, \cdot \rangle$ is the bilinear extension of the Riemannian product, and we identify $T'M$ with $\mathbb{C}^m$. Note that $\langle \mathrm{Rm}(X \wedge \overline{Y}), \overline{X \wedge \overline{Y}} \rangle = R_{X\overline{X}Y\overline{Y}}$, the bisectional curvature of the complex plane spanned by $\{X, Y\}$. Here the cones $\mathcal{C}_p$ interpolate between the cone of nonnegative bisectional curvature and that of the nonnegative curvature operator. In the next section we shall show that in fact $\mathcal{C}_p$ is an invariant condition under the Kähler-Ricci flow, which generalizes an earlier result of Bando and Mok [3, 24] on the invariance of the nonnegative bisectional curvature cone.

Let us first recall the following well-known computational lemma of Morrow and Kodaira [27]:

LEMMA 2.1 *Let ϕ be a (p,p)-form, which can be locally expressed as*

$$\phi = \frac{1}{(p!)^2} \sum \phi_{I_p, \overline{J}_p} \left(\sqrt{-1}\, dz^{i_1} \wedge dz^{\overline{j}_1}\right) \wedge \cdots \wedge \left(\sqrt{-1}\, dz^{i_p} \wedge dz^{\overline{j}_p}\right)$$

where $I_p = (i_1, \ldots, i_p)$ and $\overline{J}_p = (\overline{j}_1, \ldots, \overline{j}_p)$. Then

$$\begin{aligned}(\Delta_{\overline{\partial}}\phi)_{I_p,\overline{J}_p} = &-\frac{1}{2}\left(\sum_{ij} g^{i\overline{j}} \nabla_{\overline{j}} \nabla_i \phi_{I_p,\overline{J}_p} + \sum_{ij} g^{\overline{j}i} \nabla_i \nabla_{\overline{j}} \phi_{I_p,\overline{J}_p}\right) \\ &- \sum_{\mu=1}^{p} \sum_{\nu=1}^{p} R^{k\overline{l}}_{i_\mu\, \overline{j}_\nu} \phi_{i_1 \cdots (k)_\mu \cdots i_p, \overline{j}_1 \cdots (\overline{l})_\nu \cdots \overline{j}_p} \\ &+ \frac{1}{2}\left(\sum_{\nu=1}^{p} R^{\overline{l}}_{\overline{j}_\nu} \phi_{I_p, \overline{j}_1 \cdots (\overline{l}) \cdots \overline{j}_p} + \sum_{\mu=1}^{p} R^{k}_{i_\mu} \phi_{i_1 \cdots (k)_\mu \cdots i_p, \overline{J}_p}\right).\end{aligned} \tag{2.2}$$

Here $R_{i\overline{j}k\overline{l}}$, $R_{i\overline{j}}$, $R^{k\overline{l}}_{i_\mu \overline{j}_\nu}$, and $R^{\overline{l}}_{\overline{j}_\nu}$ are the curvature tensor, Ricci tensor, and the type-changing tensors of them via the Kähler metric, and $(k)_\mu$ means that the index in the μth position is replaced by k. Here the repeated index is summed from 1 to m.

An immediate consequence of (2.2) is that if ϕ is a solution of (2.1), then it satisfies

$$\left(\frac{\partial}{\partial t} - \Delta\right)\phi(x,t) = \mathcal{KB}(\phi) \tag{2.3}$$

where

$$\begin{aligned}(\mathcal{KB}(\phi))_{I_p,\overline{J}_p} = &\sum_{\mu=1}^{p} \sum_{\nu=1}^{p} R^{k\overline{l}}_{i_\mu \overline{j}_\nu} \phi_{i_1 \cdots (k)_\mu \cdots i_p, \overline{j}_1 \cdots (\overline{l})_\nu \cdots \overline{j}_p} \\ &- \frac{1}{2}\left(\sum_{\nu=1}^{p} R^{\overline{l}}_{\overline{j}_\nu} \phi_{I_p, \overline{j}_1 \cdots (\overline{l}) \cdots \overline{j}_p} + \sum_{\mu=1}^{p} R^{k}_{i_\mu} \phi_{i_1 \cdots (k)_\mu \cdots i_p, \overline{J}_p}\right).\end{aligned}$$

PROPOSITION 2.2 *Let (M, g) be a Kähler manifold whose curvature operator* $\mathrm{Rm} \in \mathcal{C}_p$. *Assume that $\phi(x,t)$ is a solution of* (2.1) *such that $\phi(x,0)$ is positive. Then $\phi(x,t)$ is positive for $t > 0$. If M is noncompact complete, we assume additionally that $\phi(x,t)$ satisfies an extra condition* (11.10).

PROOF. When M is a compact manifold, if we apply Hamilton's tensor maximum principle, it suffices to show that if at (x_0, t_0) there exist $v_1, \dots, v_p$ such that $\phi_{v_1 \cdots v_p, \overline{v}_1 \cdots \overline{v}_p} = 0$ and for any (x,t) with $t \le t_0$, $\phi \ge 0$, then at (x_0, t_0)

$$\mathcal{KB}(\phi)_{v_1 \cdots v_p, \overline{v}_1 \cdots \overline{v}_p} \ge 0.$$

This obviously holds if the $\{v_i\}_{i=1}^p$ are linearly dependent since $\mathcal{KB}(\phi)$ is a (p, p)-form. Hence we assume that the $\{v_i\}_{i=1}^p$ are linearly independent. By the Gramm-Schmidt process, which does not change the sign of $\phi_{v_1 \cdots v_p, \overline{v}_1 \cdots \overline{v}_p}$, we can assume that $v_1, \dots, v_p$ can be extended to a unitary frame. Hence we may assume that $(v_1, \dots, v_p) = (\frac{\partial}{\partial z_1}, \dots, \frac{\partial}{\partial z_p})$ with $(z^1, \dots, z^m)$ being a normal coordinate centered at x_0. What we need to verify is that $(\mathcal{KB}(\phi))_{1 \cdots p, \overline{1} \cdots \overline{p}} \ge 0$. Since we have that $\phi_{1 \cdots p, \overline{1} \cdots \overline{p}} = 0$ and

$$I(t) \doteqdot \phi\Big(\frac{1}{\sqrt{-1}}(v_1 + t w_1) \wedge \overline{v_1 + t w_1} \wedge \cdots \wedge \frac{1}{\sqrt{-1}}(v_p + t w_p) \wedge \overline{v_p + t w_p} \Big) \ge 0$$

for any $t \ge 0$ and any vectors $w_1, \dots, w_p$, we have $I'(0) = 0$, $I''(0) \ge 0$. The equation $I'(0) = 0$ implies that

$$\sum_{1 \le k, l \le p} \phi_{1 \cdots (w_k)_k \cdots p, \overline{1} \cdots \overline{p}} + \phi_{1 \cdots p, \overline{1} \cdots (\overline{w_l})_l \cdots \overline{p}} = 0. \tag{2.4}$$

Here $\phi_{1 \cdots (w_k)_k \cdots p, \overline{1} \cdots \overline{p}}$ means that in the kth holomorphic position, v_k is replaced by vector w_k. To simplify notation, we write $\phi_{1 \cdots (w_k)_k \cdots p, \overline{1} \cdots \overline{p}}$ as $\phi_{1 \cdots w_k \cdots p, \overline{1} \cdots \overline{p}}$. Since this holds for any p-vectors $w_1, \dots, w_p$, if we replace t by $\sqrt{-1}\, t$, one can deduce from (2.4) that

$$\sum_{1 \le k \le p} \phi_{1 \cdots w_k \cdots p, \overline{1} \cdots \overline{p}} = \sum_{1 \le l \le p} \phi_{1 \cdots p, \overline{1} \cdots \overline{w_l} \cdots \overline{p}} = 0. \tag{2.5}$$

This implies that

$$\Big(\sum_{1 \le l \le m} \sum_{\nu=1}^{p} R^{\overline{l}}_{\overline{\nu}} \phi_{1 \cdots p, \overline{1} \cdots (\overline{l})_\nu \cdots \overline{p}} + \sum_{1 \le k \le m} \sum_{\mu=1}^{p} R^{k}_{\mu} \phi_{1 \cdots (k)_\mu \cdots p, \overline{1} \cdots \overline{p}} \Big) = 0.$$

Now the fact $I''(0) \geq 0$ implies that

$$\sum_{1\leq k,l\leq p} \phi_{1\cdots w_k\cdots p,\overline{1}\cdots\overline{w_l}\cdots\overline{p}} + \sum_{1\leq k\neq l\leq p} \phi_{1\cdots w_k\cdots w_l\cdots p,\overline{1}\cdots\overline{p}} + \sum_{1\leq k\neq l\leq p} \phi_{1\cdots p,\overline{1}\cdots\overline{w_k}\cdots\overline{w_l}\cdots\overline{p}} \geq 0. \tag{2.6}$$

Replacing t by $\sqrt{-1}\,t$ in $I(t)$, the fact that $I''(0) \geq 0$ yields

$$\sum_{1\leq k,l\leq p} \phi_{1\cdots w_k\cdots p,\overline{1}\cdots\overline{w_l}\cdots\overline{p}} - \sum_{1\leq k\neq l\leq p} \phi_{1\cdots w_k\cdots w_l\cdots p,\overline{1}\cdots\overline{p}} - \sum_{1\leq k\neq l\leq p} \phi_{1\cdots p,\overline{1}\cdots\overline{w_k}\cdots\overline{w_l}\cdots\overline{p}} \geq 0. \tag{2.7}$$

Adding (2.6) and (2.7) together, we have that for any

$$\mathfrak{w} = \begin{pmatrix} w_1 \\ \vdots \\ w_p \end{pmatrix} \in \bigoplus_1^p T^{1,0}_{x_0} M,$$

the Hermitian form

$$\mathcal{J}(\mathfrak{w},\overline{\mathfrak{w}}) \doteqdot \sum_{1\leq k,l\leq p} \phi_{1\cdots w_k\cdots p,\overline{1}\cdots\overline{w_l}\cdots\overline{p}} \tag{2.8}$$

is positive semidefinite. A Hermitian bilinear form $\mathcal{J}(\mathfrak{w},\overline{\mathfrak{z}})$ can be obtained via the polarization. In matrix form, the nonnegativity of $\mathcal{J}(\,\cdot\,,\cdot\,)$ is equivalent to

$$A = \begin{pmatrix} \phi_{(\cdot)2\cdots p,\overline{(\cdot)}\overline{2}\cdots\overline{p}} & \phi_{1(\cdot)\cdots p,\overline{(\cdot)}\overline{2}\cdots\overline{p}} & \cdots & \phi_{12\cdots(\cdot)_p,\overline{(\cdot)}\overline{2}\cdots\overline{p}} \\ \phi_{(\cdot)2\cdots p,\overline{1}\overline{(\cdot)}\cdots\overline{p}} & \phi_{1(\cdot)\cdots p,\overline{1}\overline{(\cdot)}\cdots\overline{p}} & \cdots & \phi_{12\cdots(\cdot)_p,\overline{1}\overline{(\cdot)}\cdots\overline{p}} \\ \vdots & \vdots & \ddots & \vdots \\ \phi_{(\cdot)2\cdots p,\overline{12}\cdots\overline{(\cdot)}_p} & \phi_{1(\cdot)\cdots p,\overline{12}\cdots\overline{(\cdot)}_p} & \cdots & \phi_{12\cdots(\cdot)_p,\overline{12}\cdots\overline{(\cdot)}_p} \end{pmatrix}$$

is a positive semidefinite Hermitian symmetric matrix. Namely, $\overline{\mathfrak{w}}^{\mathsf{T}} A\mathfrak{w} \geq 0$. Here we view $\phi_{1\cdots(\cdot)_\mu\cdots p,\overline{1}\cdots\overline{(\cdot)}_\nu\cdots\overline{p}}$ as a matrix such that for any vectors w and z,

$$\overline{z}^{\mathsf{T}} \cdot \phi_{1\cdots(\cdot)_\mu\cdots p,\overline{1}\cdots\overline{(\cdot)}_\nu\cdots\overline{p}} \cdot w$$

is given by the Hermitian bilinear form $\phi_{1\cdots(w)_\mu\cdots p,\overline{1}\cdots\overline{(z)}_\nu\cdots\overline{p}}$. The equivalence can be made via the identity $\mathcal{J}(\mathfrak{w},\overline{\mathfrak{z}}) = \langle A(\mathfrak{w}),\overline{\mathfrak{z}}\rangle$. Here $\mathfrak{w} = w_1 \oplus \cdots \oplus w_p$, $\mathfrak{z} = z_1 \oplus \cdots \oplus z_p$. If we define $\phi^{\mu\overline{\nu}}$ by

$$\langle \phi^{\mu\overline{\nu}}(X),\overline{Y}\rangle = \phi_{1\cdots(X)_\mu\cdots p,\overline{1}\cdots\overline{(Y)}_\nu\cdots\overline{p}},$$

it is easy to see that $\overline{(\phi^{\mu\overline{\nu}})^{\mathsf{T}}} = \phi^{\nu\overline{\mu}}$. In this notation $\mathcal{J}(\mathfrak{w},\overline{\mathfrak{z}})$ or $\langle\mathcal{J}(\mathfrak{w}),\overline{\mathfrak{z}}\rangle$ can be expressed as $\sum\langle\phi^{\mu\overline{\nu}}(w_\mu),\overline{z}_\nu\rangle$. It is easy to check that $\mathcal{J}$ is Hermitian symmetric.

What must be checked is that

$$\mathcal{KB}(\phi)_{12\cdots p,\overline{12}\cdots\overline{p}} = \sum_{\mu=1}^{p}\sum_{\nu=1}^{p} R^{k\overline{l}}_{\mu\overline{\nu}}\phi_{1\cdots(k)_\mu\cdots p,\overline{1}\cdots(\overline{l})_\nu\cdots\overline{p}} \geq 0.$$

Under the unitary frame it is equivalent to

$$\sum_{\mu=1}^{p}\sum_{\nu=1}^{p} R_{\mu\overline{\nu}l\overline{k}}\phi_{1\cdots(k)_\mu\cdots p,\overline{1}\cdots(\overline{l})_\nu\cdots\overline{p}} \geq 0. \tag{2.9}$$

Here we have used the first Bianchi identity. If we can show that the Hermitian matrix

$$B = \begin{pmatrix} R_{1\overline{1}(\cdot)(\overline{\cdot})} & R_{1\overline{2}(\cdot)(\overline{\cdot})} & \cdots & R_{1\overline{p}(\cdot)(\overline{\cdot})} \\ R_{2\overline{1}(\cdot)(\overline{\cdot})} & R_{2\overline{2}(\cdot)(\overline{\cdot})} & \cdots & R_{2\overline{p}(\cdot)(\overline{\cdot})} \\ \vdots & \vdots & \ddots & \vdots \\ R_{p\overline{1}(\cdot)(\overline{\cdot})} & R_{p\overline{2}(\cdot)(\overline{\cdot})} & \cdots & R_{p\overline{p}(\cdot)(\overline{\cdot})} \end{pmatrix}$$

is nonnegative, then inequality (2.9) holds since the left-hand side of (2.9) is just the trace of the product matrix $B \cdot A$ of the two nonnegative Hermitian symmetric matrices.

On the other hand, the nonnegativity of B is equivalent to, for any $(1, 0)$-vectors $w_1, \ldots, w_p$, $\langle B(\mathfrak{w}), \overline{\mathfrak{w}}\rangle \geq 0$ with $\mathfrak{w} = w_1 \oplus \cdots \oplus w_p$. This is equivalent to

$$\sum R_{i\,\overline{j}w_j\overline{w}_i} \geq 0.$$

Let $\alpha = \sum_{i=1}^{p} \frac{\partial}{\partial z^i} \wedge \overline{w}_i$ (we later simply denote this as $\sum i \wedge \overline{w}_i$). Then the above inequality is equivalent to $\langle \mathrm{Rm}(\alpha), \overline{\alpha}\rangle \geq 0$. This proves (2.9), and hence the proposition, at least for the case when M is compact.

Defining the transformation $R^{\mu\overline{\nu}}$ as $\langle R^{\mu\overline{\nu}}(X), \overline{Y}\rangle = R_{\mu\overline{\nu}X\overline{Y}}$ provides another way to look at this. Similarly, one can easily check that $\overline{(R^{\mu\overline{\nu}})^{\mathsf{T}}} = R^{\nu\overline{\mu}}$. Define the transformation $\mathcal{K}$ on $\bigoplus_{i=1}^{p} T'M$ by $\langle \mathcal{K}(\mathfrak{w}), \overline{\mathfrak{z}}\rangle$ as $\sum\langle R^{\mu\overline{\nu}}(w_\nu), \overline{z}_\mu\rangle$. It is easy to check that $\mathcal{K}$ is Hermitian symmetric and that $\mathrm{Rm} \in \mathcal{C}_p$ implies $\mathcal{K} \geq 0$. Simple algebraic manipulation shows that

$$\begin{aligned} \text{LHS of (2.9)} &= \sum_{\mu,\nu=1}^{p} \langle R^{\mu\overline{\nu}}(e_l), \overline{e}_k\rangle\langle \phi^{\mu\overline{\nu}}(e_k), \overline{e}_l\rangle \\ &= \sum_{\mu,\nu=1}^{p} \langle R^{\mu\overline{\nu}}(\phi^{\mu\overline{\nu}}(e_k)), \overline{e}_k\rangle \end{aligned}$$

if $\{e_k\}_{k=1\cdots m}$ is a unitary frame. One can see that this is nothing but the trace of $\mathcal{K} \cdot \mathcal{J}$ since a natural unitary base for $\bigoplus_{\mu=1}^{p} T'M$ is $\{E_{\mu k}\}$, where $1 \leq \mu \leq p$,

$1 \le k \le m$, $E_{\mu k} = \vec{0} \oplus \cdots \oplus (e_k)_\mu \oplus \cdots \oplus \vec{0}$. Then $\mathcal{J}(E_{\mu k}) = \bigoplus_\nu \phi^{\mu\overline{\nu}}(e_k)$. Hence $\mathcal{K}(\mathcal{J}(E_{\mu k})) = \bigoplus_\sigma R^{\sigma\overline{\nu}}(\phi^{\mu\overline{\nu}}(e_k))$. This shows that

$$\langle \mathcal{K}(\mathcal{J}(E_{\mu k})), \overline{E_{\mu k}} \rangle = \sum \langle R^{\mu\overline{\nu}}(\phi^{\mu\overline{\nu}}(e_k)), \overline{e}_k \rangle.$$

Hence the left-hand side of (2.9) can be written as $\operatorname{trace}(\mathcal{K} \cdot \mathcal{J})$. Similarly, for any $X_1, \ldots, X_p$ we can define $R^{\mu\overline{\nu}}$, $\phi^{\mu\overline{\nu}}$, $\mathcal{K}$, and $\mathcal{J}$. The above argument shows that

$$\mathcal{K}\mathcal{B}_{X_1 \cdots X_p, \overline{X}_1 \cdots \overline{X}_p} = \operatorname{trace}(\mathcal{K} \cdot \mathcal{J}). \tag{2.10}$$

For noncompact complete manifolds, we postpone the proof to Section 11 (Theorem 11.6). □

3 Invariance of $\mathcal{C}_p$ under Kähler-Ricci Flow

Recently Wilking [41] proved a very general result on the invariance of cones of curvature operators under Ricci flow. The result is formulated for any Riemannian manifold (M, g) of real dimension n. Since our proof is a modification of his, we first state his result. Identify TM with $\mathbb{R}^n$ and its complexification $TM \otimes \mathbb{C}$ with $\mathbb{C}^n$. Also, identify $\bigwedge^2(\mathbb{R}^n)$ with the Lie algebra $\mathfrak{so}(n)$. The complexified Lie algebra $\mathfrak{so}(n, \mathbb{C})$ can be identified with $\bigwedge^2(\mathbb{C}^n)$. Its associated Lie group is $\mathsf{SO}(n, \mathbb{C})$, namely all complex matrices A satisfying $A \cdot A^{\mathsf{T}} = A^{\mathsf{T}} \cdot A = \mathrm{id}$. Recall that there exists the natural action of $\mathsf{SO}(n, \mathbb{C})$ on $\bigwedge^2(\mathbb{C}^n)$ by extending the action $g \in \mathsf{SO}(n)$ on $x \otimes y$ ($g(x \otimes y) = gx \otimes gy$). Let $\Sigma \subset \bigwedge^2(\mathbb{C}^m)$ be a subset that is invariant under the adjoint action of $\mathsf{SO}(n, \mathbb{C})$. Let $\widetilde{\mathcal{C}}_\Sigma$ be the cone of curvature operators satisfying $\langle R(v), \overline{v} \rangle \ge 0$ for any $v \in \Sigma$. Here we view the space of algebraic curvature operators as a subspace of $S^2(\bigwedge^2(\mathbb{R}^n))$ satisfying the first Bianchi identity. Wilking [41] proved the following result:

THEOREM 3.1 (Wilking) *Assume that $(M, g(t))$, for $0 \le t \le T$, is a solution of Ricci flow on a compact manifold. Assume that* $\mathrm{Rm}(g(0)) \in \widetilde{\mathcal{C}}_\Sigma$. *Then* $\mathrm{Rm}(g(t)) \in \widetilde{\mathcal{C}}_\Sigma$ *for all* $t \in [0, T]$.

By modifying the argument of Wilking, one can prove the following result:

COROLLARY 3.2 *The Kähler-Ricci flow on a compact Kähler manifold preserves the cone $\mathcal{C}_p$ for any p.*

For Riemannian manifolds, there exists another family of invariant cones that is analogous to $\mathcal{C}_p$. We say that the complex sectional curvature of (M, g) is k-positive if

$$\langle \mathrm{Rm}(v), \overline{v} \rangle > 0$$

for any nonzero $v \in \Lambda^2(\mathbb{C}^n)$ that can be written as $v = \sum_{i=1}^k Z_i \wedge W_i$ for some complex vectors Z_i and $W_i \in TM \otimes \mathbb{C}$. Clearly the complex sectional curvature being 1-positive is the same as positive complex sectional curvature. The k-positivity for $k \ge \left[\frac{n}{2}\right]$ is the same as positivity of the curvature operator. Similarly, one has the notion that the complex sectional curvature is k-nonnegative. In the

space of the algebraic curvature operators $S_B(\mathfrak{so}(n))$, the ones with k-nonnegative complex sectional curvature form a cone $\widetilde{\mathcal{C}}_k$. Clearly $\widetilde{\mathcal{C}}_k \subset \widetilde{\mathcal{C}}_{k-1}$. The argument in [36] (this in fact was first proved in an updated version of [36] in 2007) proves the following:

THEOREM 3.3 *The Ricci flow on a compact manifold preserves* $\widetilde{\mathcal{C}}_k$.

Of course, this result is now also included in the previously mentioned general theorem of Wilking. In fact, almost all the known invariant cones of nonnegativity type can be formulated as a special case of Wilking's above theorem.

Note that even for a Kähler manifold $\mathcal{C}_p$ is a bigger cone than $\widetilde{\mathcal{C}}_p$. For example, for the Kähler manifold (M, g), the nonnegativity of the complex sectional curvature implies that

$$R_{s\bar{t}i\bar{j}}(a_i\bar{b}_j - c_i\bar{d}_j)\overline{(a_t\bar{b}_s - c_t\bar{d}_s)} \geq 0$$

for any complex vector $\vec{a}$ $(= (a_1, \ldots, a_m))$, $\vec{b}$, $\vec{c}$, and $\vec{d}$. Namely, (M, g) has *strongly nonnegative sectional curvature in the sense of Siu*, which is in general stronger than the nonnegativity of the sectional curvature (or bisectional curvature). On a Kähler manifold, if $\{E_i\}$ is a unitary basis of $T'M$, and $X = a_i E_i$, $Y = b_i E_i$, $Z = d_i E_i$, and $W = c_i E_i$, then the above is equivalent to

$$\langle \mathrm{Rm}((X + \bar{Z}) \wedge (\bar{Y} + W)), \overline{(X + \bar{Z}) \wedge (\bar{Y} + W)} \rangle \geq 0.$$

In fact, a simple computation such as the above proves that $\widetilde{\mathcal{C}}_p = \mathcal{C}_{2p}$: For $1 \leq i \leq p$, let $Z_i = X_{2i-1} + \bar{Y}_{2i}$ and $W_i = \bar{Y}_{2i-1} - X_{2i}$. Then $\mathrm{Rm}(\sum_{i=1}^{p} Z_i \wedge W_i) = \mathrm{Rm}(\sum_{j=1}^{2p} X_j \wedge \bar{Y}_j)$. Thus by the fact that Rm is self-adjoint

$$\left\langle \mathrm{Rm}\left(\sum_{i=1}^{p} Z_i \wedge W_i\right), \sum_{i=1}^{p} \overline{Z_i \wedge W_i} \right\rangle = \left\langle \mathrm{Rm}\left(\sum_{j=1}^{2p} X_j \wedge \bar{Y}_j\right), \sum_{j=1}^{2p} \overline{X_j \wedge \bar{Y}_j} \right\rangle.$$

In order to prove Corollary 3.2, first let G be the subgroup of $\mathsf{SO}(n, \mathbb{C})$ consisting of matrices $A \in \mathsf{SO}(n, \mathbb{C})$ such that A commutes with the almost complex structure $J = \left(\begin{smallmatrix} 0 & \mathrm{id} \\ -\mathrm{id} & 0 \end{smallmatrix}\right)$, where we note that $n = 2m$. Let $\mathfrak{g}$ be the Lie algebra of G. A key observation is that $\mathfrak{g}$ consists of $c \in \mathfrak{so}(n, \mathbb{C})$, which commutes with J. It is easy to show that $\mathfrak{g}$ is the same as $\bigwedge^{1,1}(\mathbb{C}^m)$ under the identification of $\bigwedge^2(\mathbb{C}^n)$ with $\mathfrak{so}(n, \mathbb{C})$. More precisely, $c = (c_{ij})$ (which is identified with $c_{ij} X_i \wedge \bar{X}_j$ for a unitary basis $\{X_i\}$) is identified with $\left(\begin{smallmatrix} a & -b \\ b & a \end{smallmatrix}\right)$ with $a = c - c^{\mathsf{T}}$ and $b = -\sqrt{-1}(c + c^{\mathsf{T}})$. Now the argument of Wilking can be adapted to show the following result for the Kähler-Ricci flow.

THEOREM 3.4 *Let* $\Sigma \subset \bigwedge^{1,1}(\mathbb{C}^m)$ *be a subset invariant under the adjoint action of* G. *Let* $\mathcal{C}_\Sigma = \{\mathrm{Rm} \mid \langle \mathrm{Rm}(v), \bar{v} \rangle \geq 0$ *for any* $v \in \Sigma\}$. *Assume that* $(M, g(t))$ *(with* $t \in [0, T]$*) is a solution to the Kähler-Ricci flow on a compact Kähler manifold such that* $\mathrm{Rm}(g(0)) \in \mathcal{C}_\Sigma$. *Then for any* $t \in [0, T]$, $\mathrm{Rm}(g(t)) \in \mathcal{C}_\Sigma$.

If $C \in G$ and $v = \sum_{k=1}^{p} X_k \wedge \bar{Y}_k$ with $X_k \in T'M$ and $\bar{Y}_k \in T''M$, $C(v) = \sum_{k=1}^{p} C(X_k) \wedge C(\bar{Y}_k)$. Since C is commutative with J, $X_k' = C(X_k) \in T'M$ and $\bar{Y}_k' = C(\bar{Y}_k) \in T''M$. Hence the set consisting of all such v is an invariant set Σ_p under the adjoint action of G. The invariance of cone $\mathcal{C}_p$ follows from the above theorem by applying it to $\Sigma = \Sigma_p$.

In fact, one can easily generalize Wilking's result to manifolds with the special holonomy group. When the manifold (M, g) has a special holonomy group G with holonomy algebra $\mathfrak{g} \subset \mathfrak{so}(n)$, since for any $v \in \mathfrak{so}(n)$, $\mathrm{Rm}(v) \in \mathfrak{g}$ by the Ambrose-Singer theorem [1], Wilking's proof, in particular (A.1), remains the same even if $\{b^\alpha\}$ is an orthonormal basis of $\mathfrak{g}$ instead of an orthonormal basis of $\mathfrak{so}(n)$. Note that in this case $\mathrm{Rm}(b) = 0$ for any $b \in \mathfrak{g}^\perp$. Let $\mathfrak{g}^{\mathbb{C}} \subset \mathfrak{so}(n, \mathbb{C})$ denote the complexified Lie algebra. It is easy to see then that for any $b \in (\mathfrak{g}^{\mathbb{C}})^\perp$, $\langle \mathrm{Rm}(b), \overline{w} \rangle = \langle b, \overline{\mathrm{Rm}(w)} \rangle = 0$ for any $w \in \mathfrak{so}(n, \mathbb{C})$. Hence $\mathrm{Rm}(b) = 0$. This implies that $\langle \mathrm{Rm}^{\#}(v), \overline{w} \rangle = \frac{1}{2} \mathrm{trace}(-\mathrm{ad}_{\overline{w}} \cdot \mathrm{Rm} \cdot \mathrm{ad}_v \cdot \mathrm{Rm})$ with the trace taken for transformations of $\mathfrak{g}^{\mathbb{C}}$.

THEOREM 3.5 *Assume that (M, g_0) is a compact manifold with special holonomy group G (and corresponding Lie algebra $\mathfrak{g}$). Let Σ be a subset of $\mathfrak{g}^{\mathbb{C}}$ satisfying the assumption that it is invariant under the adjoint action of $G^{\mathbb{C}}$, the complexification of G. Then if the curvature operator* Rm *of g_0 lies inside the cone $\mathcal{C}_\Sigma$, the curvature operator* Rm *of $g(t)$, the solution to Ricci flow with initial value $g(0) = g_0$, also lies inside $\mathcal{C}_\Sigma$.*

When $G = \mathrm{U}(m)$ (with $n = 2m$), the unitary group, the above result implies Theorem 3.4. All the above results in this section remain true on noncompact manifolds if we assume that the solution $g(t)$ has bounded curvature. This shall be proved in Section 10.

4 An LYH Type Estimate for Positive (p, p)-Forms

First we recall some known computations of Morrow and Kodaira [27]. Let ϕ be a (p, q)-form valued in a holomorphic Hermitian vector bundle E with local frame $\{E_\alpha\}$ and locally $\phi = \sum \phi^\alpha E_\alpha$:

$$\phi^\alpha = \frac{1}{p!q!} \sum \phi^\alpha_{I_p \bar{J}_q} dz^{I_p} \wedge dz^{\bar{J}_q}.$$

Here $I_p = (i_1, \ldots, i_p)$, $\bar{J}_q = (\bar{j}_1, \ldots, \bar{j}_q)$, and $dz^{I_p} = dz^{i_1} \wedge \cdots \wedge dz^{i_p}$, $dz^{\bar{J}_q} = dz^{\bar{j}_1} \wedge \cdots \wedge dz^{\bar{j}_q}$. For (p, p)-forms, $\phi_{1\cdots p, \bar{1}\cdots\bar{p}}$ differs from $\phi_{1\cdots p\bar{1}\cdots\bar{p}}$ by a factor of

$$\left(\frac{1}{\sqrt{-1}}\right)^p (-1)^{\frac{p(p-1)}{2}}.$$

The following two formulae are well-known:

$$(\overline{\partial}\phi)^{\alpha}_{I_p \bar{j}_0 \cdots \bar{j}_q} = (-1)^p \sum_{\nu=0}^{q} (-1)^{\nu} \nabla_{\bar{j}_\nu} \phi^{\alpha}_{I_p \bar{j}_0 \cdots \hat{\bar{j}}_\nu \cdots \bar{j}_q}, \tag{4.1}$$

$$(\overline{\partial}^* \phi)^{\alpha}_{I_p \bar{j}_1 \cdots \bar{j}_{q-1}} = (-1)^{p+1} \sum_{ij} g^{\bar{j}i} \nabla_i \phi^{\alpha}_{I_p \overline{ij j_1} \cdots \bar{j}_{q-1}}. \tag{4.2}$$

Here $\hat{\bar{j}}_\nu$ means that the index $\bar{j}_\nu$ is removed. From (4.1) and (4.2) we have

$$\begin{aligned} (\Delta_{\overline{\partial}}\phi)^{\alpha}_{I_p \bar{J}_q} = &- \sum_{ij} g^{\bar{j}i} \nabla_i \nabla_{\bar{j}} \phi^{\alpha}_{I_p \bar{J}_q} + \sum_{\nu=1}^{q} \Omega^{\alpha \bar{l}}_{\beta\, \bar{j}_\nu} \phi^{\beta}_{I_p \bar{j}_1 \cdots (\bar{l})_\nu \cdots \bar{j}_q} \\ &+ \sum_{\nu=1}^{q} R^{\bar{l}}_{\bar{j}_\nu} \phi^{\alpha}_{I_p \bar{j}_1 \cdots (\bar{l}) \cdots \bar{j}_q} - \sum_{\mu=1}^{p} \sum_{\nu=1}^{q} R^{k\bar{l}}_{i_\mu\, \bar{j}_\nu} \phi^{\alpha}_{i_1 \cdots (k)_\mu \cdots i_p \bar{j}_1 \cdots (\bar{l})_\nu \cdots \bar{j}_q} \end{aligned} \tag{4.3}$$

and

$$\begin{aligned} (\Delta_{\overline{\partial}}\phi)^{\alpha}_{I_p \bar{J}_q} = &- \sum_{ij} g^{i\bar{j}} \nabla_{\bar{j}} \nabla_i \phi^{\alpha}_{I_p \bar{J}_q} + \sum_{\nu=1}^{q} \Omega^{\alpha \bar{l}}_{\beta\, \bar{j}_\nu} \phi^{\beta}_{I_p \bar{j}_1 \cdots (\bar{l})_\nu \cdots \bar{j}_q} - \sum_{\beta} \Omega^{\alpha}_{\beta} \phi^{\beta}_{I_p \bar{J}_q} \\ &+ \sum_{\mu=1}^{p} R^{k}_{i_\mu} \phi^{\alpha}_{i_1 \cdots (k)_\mu \cdots i_p \bar{J}_q} - \sum_{\mu=1}^{p} \sum_{\nu=1}^{q} R^{k\bar{l}}_{i_\mu \bar{j}_\nu} \phi^{\alpha}_{i_1 \cdots (k)_\mu \cdots i_p \bar{j}_1 \cdots (\bar{l})_\nu \cdots \bar{j}_q}, \end{aligned} \tag{4.4}$$

where $(k)_\mu$ means that the index in the μth position is replaced by k. Here

$$\Theta^{\alpha}_{\beta} = \frac{\sqrt{-1}}{2\pi} \sum \Omega^{\alpha}_{\beta i \bar{j}}\, dz^i \wedge d\bar{z}^j$$

is the curvature of E and Ω^{α}_{β} is the mean curvature. Here we shall focus on the case where E is the trivial bundle.

Recall the contraction Λ operator, $\Lambda : \bigwedge^{p,q} \to \bigwedge^{p-1,q-1}$, defined by

$$(\Lambda\phi)_{i_1 \cdots i_{p-1} \bar{j}_1 \cdots \bar{j}_{q-1}} = \frac{1}{\sqrt{-1}} (-1)^{p-1} g^{i\bar{j}} \phi_{i i_1 \cdots i_{p-1} \bar{j} \bar{j}_1 \cdots \bar{j}_{q-1}}.$$

Note our definition of Λ differs from [27] by a sign. With the above notation the Kähler identities assert

$$\partial\Lambda - \Lambda\partial = -\sqrt{-1}\,\overline{\partial}^*, \quad \overline{\partial}\Lambda - \Lambda\overline{\partial} = \sqrt{-1}\,\partial^*. \tag{4.5}$$

In [33], the first author speculated that for ϕ, a nonnegative (p, p)-form satisfying the Hodge Laplacian heat equation, the $(p-1, p-1)$-form

$$\begin{aligned} Q(\phi, V) \doteqdot &\frac{1}{2\sqrt{-1}} (\overline{\partial}^* \partial^* - \partial^* \overline{\partial}^*)\phi + \frac{1}{\sqrt{-1}} (\overline{\partial}^* \phi)_V \\ &- \frac{1}{\sqrt{-1}} (\partial^* \phi)_{\overline{V}} + \phi_{V, \overline{V}} + \frac{\Lambda\phi}{t} \geq 0 \end{aligned} \tag{4.6}$$

for any $(1,0)$-type vector field V. Here $\phi_{V,\overline{V}}$ is a $(p-1, p-1)$-form defined as

$$(\phi_{V,\overline{V}})_{I_{p-1}, \bar{J}_{p-1}} \doteqdot \phi_{V I_{p-1}, \overline{V}\, \bar{J}_{p-1}}$$

or equivalently

$$(\phi_{V,\overline{V}})_{I_{p-1}\overline{J}_{p-1}} \doteqdot \frac{1}{\sqrt{-1}}(-1)^{p-1}\phi_{VI_{p-1}\overline{V}\,\overline{J}_{p-1}},$$

and for ψ and ψ', ψ_V and $\psi'_{\overline{V}}$ are defined as

$$(\psi_V)_{I_{p-1}\overline{J}_q} \doteqdot \psi_{VI_{p-1}\overline{J}_q}, \quad (\psi'_{\overline{V}})_{I_p\overline{J}_{q-1}} \doteqdot (-1)^p\psi'_{I_p\overline{V}\overline{J}_{q-1}}.$$

When the meaning is clear we abbreviate $Q(\phi, V)$ as Q. The expression of Q coincides with the quantity Z in theorem 1.1 of [30] for $(1,1)$-forms due to (4.1), (4.2) as well as their cousins

$$(\partial\phi)^{\alpha}_{i_0i_1\cdots i_p\overline{J}_q} = \sum_{\mu=0}^{p}(-1)^{\mu}\nabla_{i_\mu}\phi^{\alpha}_{i_0\cdots\hat{i}_\mu\cdots i_p\overline{J}_q}, \tag{4.7}$$

$$(\partial^*\phi)^{\alpha}_{i_1\cdots i_{p-1}\overline{J}_q} = -\sum_{ij} g^{i\overline{j}}\nabla_{\overline{j}}\phi^{\alpha}_{ii_1\cdots i_{p-1}\overline{J}_q} \tag{4.8}$$

since the operators (defined in [30] for $p = 1$)

$$\operatorname{div}(\phi)_{i_1\cdots i_{p-1},\overline{J}_p}, \quad \operatorname{div}(\phi)_{I_p,\overline{j}_1\cdots\overline{j}_{p-1}}, \quad \text{and} \quad g^{i\overline{j}}\nabla_i \operatorname{div}(\phi)_{I_{p-1},\overline{j}\,\overline{j}_1\cdots\overline{j}_{p-1}},$$

can be identified with ∂^*, $\overline{\partial}^*$, and $\overline{\partial}^*\partial^*$, etc. To make it precise, first note that in our discussion the bundle is trivial and we can forget about the upper index in ϕ^{α}. It is easy to check that

$$\Lambda(\phi)_{I_{p-1},\overline{J}_{p-1}} = g^{i\overline{j}}\phi_{iI_{p-1},\overline{j}\overline{J}_{p-1}}.$$

In (4.6), $(\overline{\partial}^*\phi)_V$ is a $(p-1, p-1)$-form, which by definition can be written as

$$(\overline{\partial}^*\phi)_V = \sum_{i=1}^{m} V^i\iota_i(\overline{\partial}^*\phi)$$

where ι_i is the adjoint of the operator $dz^i \wedge (\cdot)$. Hence $(\overline{\partial}^*\phi)_V = \iota_V \cdot \overline{\partial}^*\phi$. A direct calculation then shows that

$$\frac{1}{\sqrt{-1}}((\overline{\partial}^*\phi)_V)_{I_{p-1},\overline{J}_{p-1}} = V^i g^{\overline{j}k}\nabla_k\phi_{iI_{p-1},\overline{j}\overline{J}_{p-1}}.$$

Similarly, $(\partial^*\phi)_{\overline{V}} = \sum_{j=1}^{m} V^{\overline{j}}\iota_{\overline{j}}(\partial^*\phi) = \iota_{\overline{V}}\cdot\partial^*\phi$ and another direct computation implies that

$$\frac{1}{\sqrt{-1}}((\partial^*\phi)_{\overline{V}})_{I_{p-1},\overline{J}_{p-1}} = -\overline{V^j}\,g^{i\overline{k}}\nabla_{\overline{k}}\phi_{iI_{p-1},\overline{j}\overline{J}_{p-1}}.$$

If we define

$$(\mathrm{div}''(\phi))_{I_p,\overline{J}_{p-1}} \doteqdot \sum_{ij} g^{\overline{j}i}\nabla_i \phi_{I_p,\overline{jJ}_{p-1}},$$

$$(\mathrm{div}'(\phi))_{I_{p-1},\overline{J}_p} \doteqdot \sum_{ij} g^{i\overline{j}}\nabla_{\overline{j}} \phi_{iI_{p-1},\overline{J}_p},$$

$$(\mathrm{div}''_V(\phi))_{I_{p-1},\overline{J}_{p-1}} \doteqdot (\mathrm{div}''(\phi))_{VI_{p-1},\overline{J}_{p-1}},$$

$$(\mathrm{div}'_{\overline{V}}(\phi))_{I_{p-1},\overline{J}_{p-1}} \doteqdot (\mathrm{div}'(\phi))_{I_{p-1},\overline{V}\overline{J}_{p-1}},$$

then a simple calculation shows that

$$\frac{1}{\sqrt{-1}}(\overline{\partial}^*\phi)_V = \mathrm{div}''_V(\phi), \quad \frac{1}{\sqrt{-1}}(\partial^*\phi)_{\overline{V}} = -\mathrm{div}'_{\overline{V}}(\phi),$$
$$(\overline{\partial}^*\partial^*\phi)_{I_{p-1},\overline{J}_{p-1}} = \sqrt{-1}\,\mathrm{div}''(\mathrm{div}'(\phi))_{I_{p-1},\overline{J}_{p-1}},$$
$$(\partial^*\overline{\partial}^*\phi)_{I_{p-1},\overline{J}_{p-1}} = -\sqrt{-1}\,\mathrm{div}'(\mathrm{div}''(\phi))_{I_{p-1},\overline{J}_{p-1}}.$$

Hence Q also has the following equivalent form:

$$\begin{aligned} Q_{I_{p-1},\overline{J}_{p-1}} = {} & \frac{1}{2}\big[\mathrm{div}''(\mathrm{div}'(\phi)) + \mathrm{div}'(\mathrm{div}''(\phi))\big]_{I_{p-1},\overline{J}_{p-1}} \\ & + (\mathrm{div}'(\phi))_{I_{p-1},\overline{V}\overline{j}_1\cdots\overline{j}_{p-1}} + (\mathrm{div}''(\phi))_{Vi_1\cdots i_{p-1},\overline{J}_{p-1}} \\ & + \phi_{VI_{p-1},\overline{V}\overline{J}_{p-1}} + \frac{1}{t}(\Lambda\phi)_{I_{p-1},\overline{J}_{p-1}}. \end{aligned}$$

Recall that $d = \partial + \overline{\partial}$ and $d_c = -\sqrt{-1}(\partial - \overline{\partial})$. One can also write Q as

$$Q = \frac{1}{2}\, d_c^* d^*\phi - \Pi_{p-1,p-1}\cdot \iota_{V+\overline{V}}\cdot d_c^*\phi + \phi_{V,\overline{V}} + \frac{\Lambda\phi}{t} \tag{4.9}$$

as well as

$$Q = \frac{1}{\sqrt{-1}}\,\overline{\partial}^*\partial^*\phi + \frac{1}{\sqrt{-1}}\,\iota_V\cdot\overline{\partial}^*\phi - \frac{1}{\sqrt{-1}}\,\iota_{\overline{V}}\cdot\partial^*\phi + \phi_{V,\overline{V}} + \frac{\Lambda\phi}{t}. \tag{4.10}$$

Here $\Pi_{p-1,p-1}$ is the projection to the $\bigwedge^{p-1,p-1}$-space. When ϕ is d-closed and we write $\psi = \Lambda\phi$, the Kähler identities and its consequence $\Delta_{\overline{\partial}} = \Delta_\partial$ imply that

$$Q = \psi_t + \frac{1}{2}(\overline{\partial\partial}^* + \partial\partial^*)\psi + \iota_V\cdot\partial\psi + \iota_{\overline{V}}\cdot\overline{\partial}\psi + \phi_{V,\overline{V}} + \frac{\psi}{t}.$$

Sometimes we also abbreviate $\iota_V\partial\psi, \iota_{\overline{V}}\overline{\partial}\psi$ as $\partial_V\psi, \overline{\partial}_{\overline{V}}\psi$.

THEOREM 4.1 *Let (M, g) be a complete Kähler manifold. Assume that $\phi(x,t)$ is a positive solution to* (2.1)*. Assume further that the curvature of (M, g) satisfies $\mathcal{C}_p$. Then $Q \geq 0$ for any $t > 0$. Furthermore, if the equality holds somewhere for $t > 0$, then (M, g) must be flat.*

PROOF. We postpone the proof on the part that $Q \geq 0$ to Sections 5 and 11. Here we show the rigidity result implied by the equality case, which can be reduced to the $p = 1$ case treated in [30]. The observation is that $\Lambda(Q(\phi, V)) = Q(\Lambda\phi, V)$. This can be seen via the well-known facts (cf. [40, chap. 5, cor. 4.10]) that $[\Lambda, \partial^*] = [\Lambda, \overline{\partial}^*] = 0$ as well as

$$\phi_{V,\overline{V}} = \sqrt{-1} \sum_{ij=1}^{m} V^i V^{\overline{j}} \iota_i \iota_{\overline{j}} \phi = \sqrt{-1}\, \iota_V \cdot \iota_{\overline{V}} \cdot \phi$$

and the equalities $[\Lambda, \iota_V] = [\Lambda, \iota_{\overline{V}}] = 0$. (See [40, chap. 5, (3.19)] for a proof of the above identities.) Hence if $Q(\phi, V) = 0$, it implies that $Q(\Lambda^{p-1}\phi, V) = \Lambda^{p-1}(Q(\phi, V)) = 0$. Now the result follows from [30, theorem 1.1] applied to $\Lambda^{p-1}\phi$, which is a positive $(1, 1)$-form. □

Note that in the statement of the theorem, $Q(\phi, V) = 0$ means it equals 0 as a $(p-1, p-1)$-form.

COROLLARY 4.2 *Let (M, g) and ϕ be as above. Let $\psi = \Lambda\phi$ and assume that ϕ is d-closed. Then*

$$\frac{1}{t}\frac{\partial}{\partial t}(t\psi) + \frac{1}{2}(\overline{\partial\overline{\partial}}^* + \partial\partial^*)\psi \geq -\min_V(\partial_V\psi + \overline{\partial}_{\overline{V}}\psi + \phi(V, \overline{V})) \geq 0. \tag{4.11}$$

In particular, for any $\phi \geq 0$, if M is compact,

$$\frac{d}{dt}\int_M t\psi \wedge \omega^{m-p+1} \geq 0. \tag{4.12}$$

PROOF. By adding $\epsilon\omega^p$ and then letting $\epsilon \to 0$, we can assume that ϕ is strictly positive. Then for any $I_{p-1} = (i_1, \ldots, i_{p-1})$, the Hermitian bilinear form (in V),

$$\left(\psi_t + \frac{\psi}{t}\right)_{I_{p-1},\overline{I}_{p-1}} + (\partial_V\psi + \overline{\partial}_{\overline{V}}\psi)_{I_{p-1},\overline{I}_{p-1}} + \phi(V, \overline{V})_{I_{p-1},\overline{I}_{p-1}}$$

has a minimum. It is then easy to see that for the minimizing vector V,

$$(\partial_V\psi + \overline{\partial}_{\overline{V}}\psi)_{I_{p-1},\overline{I}_{p-1}} + \phi(V, \overline{V})_{I_{p-1},\overline{I}_{p-1}} = -\phi(V, \overline{V})_{I_{p-1},\overline{I}_{p-1}}.$$

The first result then follows. For the second one, just notice that

$$\int_M (\overline{\partial\overline{\partial}}^* + \partial\partial^*)\psi \wedge \omega^{m-p+1} = 0.$$

□

Remark 4.3. Clearly, if one can perform the integration by parts and control the boundary terms, the monotonicity (4.12) still holds in the noncompact case.

One can define a formal dual operator of $Q(\phi, V)$ as

$$\begin{aligned} Q^*(\psi, V^*) \doteq\ & \sqrt{-1}\partial\overline{\partial}\psi + \sqrt{-1}V^* \wedge \overline{\partial}\psi - \sqrt{-1}\,\overline{V}^* \wedge \partial\psi \\ & + \sqrt{-1}V^* \wedge \overline{V}^* \wedge \psi + \frac{\omega \wedge \psi}{t}, \end{aligned} \tag{4.13}$$

which maps a $(m-p, m-p)$-form ψ to a $(m-p+1, m-p+1)$-form. Here V^* is a $(1,0)$-type covector. The following duality can be checked by direct calculations by making use of the following well-known identities on (p,q)-forms (cf. proposition 2.4, (1.13), and (3.14), all in chapter 5 in [40], respectively):

$$\overline{\partial}^* = -*\cdot\partial\cdot *, \qquad \partial^* = -*\cdot\overline{\partial}\cdot *;$$

$$\Lambda = (-)^{p+q} *\cdot(\omega\wedge)\cdot *;$$

$$\iota_V = *\cdot(\overline{V}^*\wedge)\cdot *, \quad \iota_{\overline{V}} = *\cdot(V^*\wedge)\cdot *.$$

Here $*$ is the Hodge star operator.

PROPOSITION 4.4 *Let ϕ and V be as in the above discussion. Let V^* be the dual of V, and let $*$ be the Hodge star operator. Then*

$$Q(\phi, V) = *\cdot Q^*(*\cdot\phi, V^*). \tag{4.14}$$

By this duality, one can identify the result for the (m,m)-forms with that of [9]. In the rest of this section we derive some preliminary results useful for the proof of Theorem 4.1. The following lemma follows from (4.3), (4.4), and the fact that $[\Delta_{\overline{\partial}}, \partial^*] = [\Delta_{\overline{\partial}}, \overline{\partial}^*] = 0$.

LEMMA 4.5 *Let ϕ be a (p,p)-form satisfying* (2.1). *Then* $(\frac{\partial}{\partial t} + \Delta_{\overline{\partial}})\overline{\partial}^*\phi = (\frac{\partial}{\partial t} + \Delta_{\overline{\partial}})\partial^*\phi = 0$. *Hence by* (4.3) *and* (4.4)

$$\begin{aligned}
&\left(\frac{\partial}{\partial t} - \Delta\right)(\overline{\partial}^*\phi)_{ii_1\cdots i_{p-1},\overline{j}_1\cdots\overline{j}_{p-1}} \\
&\quad= \sum_{\mu=1}^{p-1}\sum_{\nu=1}^{p-1} R^{k\overline{l}}_{i_\mu\ \overline{j}_\nu}(\overline{\partial}^*\phi)_{ii_1\cdots(k)_\mu\cdots i_{p-1},\overline{j}_1\cdots(\overline{l})_\nu\cdots\overline{j}_{p-1}} \\
&\quad\quad+ \sum_{\nu=1}^{p-1} R^{k\overline{l}}_{i\ \overline{j}_\nu}(\overline{\partial}^*\phi)_{ki_1\cdots i_{p-1},\overline{j}_1\cdots(\overline{l})_\nu\cdots\overline{j}_{p-1}} - \frac{1}{2}R^k_i(\overline{\partial}^*\phi)_{ki_1\cdots i_{p-1},\overline{j}_1\cdots\overline{j}_{p-1}} \\
&\quad\quad- \frac{1}{2}\left(\sum_{\nu=1}^{p-1} R^{\overline{l}}_{\overline{j}_\nu}(\overline{\partial}^*\phi)_{ii_1\cdots i_{p-1},\overline{j}_1\cdots(\overline{l})_\nu\cdots\overline{j}_{p-1}}\right. \\
&\quad\quad\quad\left.+ \sum_{\mu=1}^{p-1} R^k_{i_\mu}(\overline{\partial}^*\phi)_{ii_1\cdots(k)_\mu\cdots i_{p-1},\overline{j}_1\cdots\overline{j}_{p-1}}\right),
\end{aligned} \tag{4.15}$$

$$\begin{aligned}
&\left(\frac{\partial}{\partial t} - \Delta\right)(\partial^*\phi)_{i_1\cdots i_{p-1},\overline{j}\,\overline{j}_1\cdots\overline{j}_{p-1}} \\
&\quad= \sum_{\mu=1}^{p-1}\sum_{\nu=1}^{p-1} R^{k\overline{l}}_{i_\mu\ \overline{j}_\nu}(\partial^*\phi)_{i_1\cdots(k)_\mu\cdots i_{p-1},\overline{j}\,\overline{j}_1\cdots(\overline{l})_\nu\cdots\overline{j}_{p-1}} \\
&\quad\quad+ \sum_{\mu=1}^{p-1} R^{k\overline{l}}_{i_\mu\ \overline{j}}(\partial^*\phi)_{i_1\cdots(k)_\mu\cdots i_{p-1},\overline{l}\,\overline{j}_1\cdots\overline{j}_{p-1}} - \frac{1}{2}R^{\overline{l}}_{\overline{j}}(\partial^*\phi)_{i_1\cdots i_{p-1},\overline{l}\,\overline{j}_1\cdots\overline{j}_{p-1}} -
\end{aligned} \tag{4.16}$$

$$- \frac{1}{2}\Bigg(\sum_{\nu=1}^{p-1} R^{\bar{l}}_{\bar{j}_\nu}(\partial^*\phi)_{i_1\cdots i_{p-1},\bar{j}\bar{j}_1\cdots(\bar{l})_\nu\cdots\bar{j}_{p-1}} + \sum_{\mu=1}^{p-1} R^{k}_{i_\mu}(\partial^*\phi)_{i_1\cdots(k)_\mu\cdots i_{p-1},\bar{j}\bar{j}_1\cdots\bar{j}_{p-1}}\Bigg).$$

Similarly, one can write the following lemma:

LEMMA 4.6 *Let ϕ be a solution to* (2.1)*. Then*

$$\left(\frac{\partial}{\partial t} + \Delta_{\bar{\partial}}\right)(\partial^*\bar{\partial}^*\phi) = \left(\frac{\partial}{\partial t} + \Delta_{\bar{\partial}}\right)(\bar{\partial}^*\partial^*\phi) = 0.$$

Hence (4.3) *and* (4.4) *imply the following equations for* $(\frac{\partial}{\partial t} - \Delta)(\partial^*\bar{\partial}^*\phi)$:

$$\begin{aligned}
&\left(\frac{\partial}{\partial t} - \Delta\right)(\partial^*\bar{\partial}^*\phi)_{i_1\cdots i_{p-1},\bar{j}_1\cdots\bar{j}_{p-1}} \\
&\quad= \sum_{\mu=1}^{p-1}\sum_{\nu=1}^{p-1} R^{k\bar{l}}_{i_\mu\ \bar{j}_\nu}(\partial^*\bar{\partial}^*\phi)_{i_1\cdots(k)_\mu\cdots i_{p-1},\bar{j}_1\cdots(\bar{l})_\nu\cdots\bar{j}_{p-1}} \\
&\quad\quad- \frac{1}{2}\Bigg(\sum_{\nu=1}^{p-1} R^{\bar{l}}_{\bar{j}_\nu}(\partial^*\bar{\partial}^*\phi)_{i_1\cdots i_{p-1},\bar{j}_1\cdots(\bar{l})_\nu\cdots\bar{j}_{p-1}} \\
&\quad\quad\quad+ \sum_{\mu=1}^{p-1} R^{k}_{i_\mu}(\partial^*\bar{\partial}^*\phi)_{i_1\cdots(k)_\mu\cdots i_{p-1},\bar{j}_1\cdots\bar{j}_{p-1}}\Bigg).
\end{aligned} \tag{4.17}$$

Similar equations can be written for $(\frac{\partial}{\partial t} - \Delta)(\bar{\partial}^*\partial^*\phi)$*. Simply put,*

$$\left(\frac{\partial}{\partial t} - \Delta\right)(\partial^*\bar{\partial}^*\phi) = \mathcal{KB}(\partial^*\bar{\partial}^*\phi) \quad \textit{and} \quad \left(\frac{\partial}{\partial t} - \Delta\right)(\bar{\partial}^*\partial^*\phi) = \mathcal{KB}(\bar{\partial}^*\partial^*\phi).$$

Lemma 4.5 implies that

$$\begin{aligned}
&\left(\frac{\partial}{\partial t} - \Delta\right)(\mathrm{div}''_V(\phi))_{I_{p-1},\overline{J}_{p-1}} \\
&\quad= \mathrm{div}''_{(\frac{\partial}{\partial t}-\Delta)V}(\phi) + \mathcal{KB}(\mathrm{div}''_V(\phi)) \\
&\quad\quad+ \sum_{\nu=1}^{p-1} R^{k\bar{l}}_{V\ \bar{j}_\nu}(\mathrm{div}''(\phi))_{ki_1\cdots i_{p-1},\bar{j}_1\cdots(\bar{l})_\nu\cdots\bar{j}_{p-1}} - \frac{1}{2}\,\mathrm{div}''_{\mathrm{Ric}(V)}(\phi) \\
&\quad\quad- g^{i\bar{j}}\big((\nabla_i\,\mathrm{div}''(\phi))_{\nabla_{\bar{j}}V} + (\nabla_{\bar{j}}\,\mathrm{div}''(\phi))_{\nabla_i V}\big);
\end{aligned} \tag{4.18}$$

$$
\begin{aligned}
&\left(\frac{\partial}{\partial t}-\Delta\right)(\operatorname{div}'_{\overline{V}}(\phi))_{I_{p-1},\bar{J}_{p-1}}\\
&\quad=\operatorname{div}'_{(\frac{\partial}{\partial t}-\Delta)\overline{V}}(\phi)+\mathcal{KB}(\operatorname{div}'_{\overline{V}}(\phi))\\
&\qquad+\sum_{\mu=1}^{p-1}R^{k\bar{l}}_{i_\mu\,\bar{V}}(\operatorname{div}'(\phi))_{i_1\cdots(k)_\mu\cdots i_{p-1},\bar{l}\bar{j}_1\cdots\bar{j}_{p-1}}-\frac{1}{2}\operatorname{div}'_{\overline{\mathrm{Ric}(V)}}(\phi)\\
&\qquad-g^{i\bar{j}}\big((\nabla_i\operatorname{div}'(\phi))_{\nabla_{\bar{j}}\overline{V}}+(\nabla_{\bar{j}}\operatorname{div}'(\phi))_{\nabla_i\overline{V}}\big),
\end{aligned}
\tag{4.19}
$$

since for any r-tensor T, the $(r-1)$–tensor

$$T_V(X_1,\dots X_{r-1})\doteqdot T(V,X_1,\dots X_{r-1})$$

satisfies $\nabla_X T_V=(\nabla_X T)_V+T_{\nabla_X V}$. To compute the evolution equation of Q, since $(\frac{\partial}{\partial t}-\Delta)\Lambda\phi=\mathcal{KB}(\Lambda\phi)$, the only term left is the evolution equation on $\phi(V,\overline{V})$, which we also abbreviate as $\phi_{V,\bar{V}}$. Since

$$\nabla_X\phi_{V,\bar{V}}=(\nabla_X\phi)_{V,\bar{V}}+\phi_{\nabla_X V,\bar{V}}+\phi_{V,\nabla_X\bar{V}},$$

Lemma 2.1 implies that

$$
\begin{aligned}
&\left(\frac{\partial}{\partial t}-\Delta\right)\phi_{V,\bar{V}}\\
&\quad=\mathcal{KB}(\phi_{V,\bar{V}})+R^{k\bar{l}}_{V\,\bar{V}}\phi_{k,\bar{l}}+\sum_{\nu=1}^{p-1}R^{k\bar{l}}_{V\,\bar{j}_\nu}\phi_{kI_{p-1},\bar{V}\bar{j}_1\cdots(\bar{l})_\nu\cdots\bar{j}_{p-1}}\\
&\qquad+\sum_{\mu=1}^{p-1}R^{k\bar{l}}_{i_\mu\,\bar{V}}\phi_{Vi_1\cdots(k)_\mu\cdots i_{p-1},\bar{l}\bar{J}_{p-1}}-\frac{1}{2}\big(\phi_{V,\overline{\mathrm{Ric}(V)}}+\phi_{\mathrm{Ric}(V),\bar{V}}\big)\\
&\qquad+\phi_{(\frac{\partial}{\partial t}-\Delta)V,\bar{V}}+\phi_{V,(\frac{\partial}{\partial t}-\Delta)\bar{V}}-g^{i\bar{j}}(\phi_{\nabla_i V,\nabla_{\bar{j}}\bar{V}}+\phi_{\nabla_{\bar{j}}V,\nabla_i\bar{V}})\\
&\qquad-g^{i\bar{j}}\big((\nabla_i\phi)_{\nabla_{\bar{j}}V,\bar{V}}+(\nabla_i\phi)_{V,\nabla_{\bar{j}}\bar{V}}+(\nabla_{\bar{j}}\phi)_{\nabla_i V,\bar{V}}+(\nabla_{\bar{j}}\phi)_{V,\nabla_i\bar{V}}\big).
\end{aligned}
\tag{4.20}
$$

5 Proof of Theorem 1.1

Now Q is viewed as a $(p-1,p-1)$-form. For $p=1$ and $p=m$, the result has been proven earlier. Using the notation introduced in the last section, the LYH quantity Q, a $(p-1,p-1)$-form depending on vector field V, can be written as

$$Q=\frac{1}{2}\big[\operatorname{div}''(\operatorname{div}'(\phi))+\operatorname{div}'(\operatorname{div}''(\phi))\big]+\operatorname{div}'_{\overline{V}}(\phi)+\operatorname{div}''_{V}(\phi)+\phi_{V,\overline{V}}+\frac{\Lambda\phi}{t}.$$

As before if we assume that at (x_0,t_0), for the first time, for some V,

$$Q_{v_1\cdots v_{p-1},\bar{v}_1\cdots\bar{v}_{p-1}}=0$$

for some linearly independent vectors $\{v_i\}_{i=1}^{p-1}$, then we shall prove

$$\left[\left(\frac{\partial}{\partial t}-\Delta\right)Q\right]_{v_1\cdots v_{p-1},\overline{v}_1\cdots\overline{v}_{p-1}}\geq 0.$$

By a perturbation argument as in [30], we can assume without loss of generality that ϕ is strictly positive. As in [17], it suffices to check that at the point (x_0,t_0), $(\frac{\partial}{\partial t}-\Delta)Q\geq 0$. Since the complex function (in terms of the variable z)

$$\begin{aligned}I(z)\doteq{}&\frac{1}{2}\left[\mathrm{div}''(\mathrm{div}'(\phi))+\mathrm{div}'(\mathrm{div}''(\phi))\right]_{v_1(z)\cdots v_{p-1}(z),\overline{v}_1(z)\cdots\overline{v}_{p-1}(z)}\\&+[\mathrm{div}'_{\overline{V}(z)}(\phi)+\mathrm{div}''_{V(z)}(\phi)]_{v_1(z)\cdots v_{p-1}(z),\overline{v}_1(z)\cdots\overline{v}_{p-1}(z)}\\&+\phi_{V(z)v_1(z)\cdots v_{p-1}(z),\overline{V}(z)\overline{v}_1(z)\cdots\overline{v}_{p-1}(z)}+\frac{\Lambda\phi_{v_1(z)\cdots v_{p-1}(z),\overline{v}_1(z)\cdots\overline{v}_{p-1}(z)}}{t}\end{aligned}$$

satisfies $I(0)=0$ and $I(z)\geq 0$ for any variational vectors $v_\mu(z)$ and $V(z)$, holomorphic in z, with $v_\mu(0)=v_\mu$ and $V(0)=V$, we have $I_z(0)=I_{\overline{z}}(0)=0$ and $I_{z\overline{z}}(0)\geq 0$. In particular, letting $v_\mu(z)=v_\mu$ and $V'(0)=X$, we have that

$$\mathrm{div}'_{\overline{X}}(\phi)+\phi_{V,\overline{X}}=0=\mathrm{div}''_X(\phi)+\phi_{X,\overline{V}}. \tag{5.1}$$

Similarly, by fixing $V(z)=V$ and varying $v_\mu(z)$, we deduce for any X,

$$Q_{v_1\cdots(X)_\mu\cdots v_{p-1},\overline{v}_1\cdots\overline{v}_{p-1}}=0=Q_{v_1\cdots v_{p-1},\overline{v}_1\cdots(\overline{X})_\nu\cdots v_{p-1}}. \tag{5.2}$$

As before, after a change of variables, we may assume that $\{v_i\}_{i=1}^{p-1}=\{\frac{\partial}{\partial z^i}\}_{i=1}^{p-1}$. Since $z=0$ is the minimizing point, we have that $\Delta I(0)\geq 0$. If $v'_\mu(0)=X_\mu$ and $V'(0)=X$, where $v'(z)=\frac{\partial v}{\partial z}$, then

$$\begin{aligned}&\sum_{\mu,\nu=1}^{p-1}Q_{v_1\cdots X_\mu\cdots v_{p-1},\overline{v}_1\cdots\overline{X}_\nu\cdots\overline{v}_{p-1}}+\phi_{Xv_1\cdots v_{p-1},\overline{X}\overline{v}_1\cdots\overline{v}_{p-1}}\\&\quad+\sum_{\mu=1}^{p-1}\mathrm{div}'_{\overline{X}}(\phi)_{v_1\cdots X_\mu\cdots v_{p-1},\overline{v}_1\cdots\overline{v}_{p-1}}+\phi_{Vv_1\cdots X_\mu\cdots v_{p-1},\overline{X}\overline{v}_1\cdots\overline{v}_{p-1}}\\&\quad+\sum_{\nu=1}^{p-1}\mathrm{div}''_X(\phi)_{v_1\cdots v_{p-1},\overline{v}_1\cdots\overline{X}_\nu\cdots\overline{v}_{p-1}}+\phi_{Xv_1\cdots v_{p-1},\overline{V}\overline{v}_1\cdots\overline{X}_\nu\cdots\overline{v}_{p-1}}\geq 0.\end{aligned}$$

This implies that the block matrix

$$\mathcal{M}_1=\begin{pmatrix}A & S\\ \overline{S}^{\mathsf{T}} & \phi_{(\cdot)1\cdots p-1,(\cdot)\overline{1}\cdots\overline{p-1}}\end{pmatrix}\geq 0$$

where

$$A = \begin{pmatrix} Q_{(\cdot)2\cdots p-1,\overline{(\cdot)}\overline{2}\cdots\overline{p-1}} & Q_{1(\cdot)\cdots p-1,\overline{(\cdot)}\overline{2}\cdots\overline{p-1}} & \cdots & Q_{12\cdots(\cdot)_{p-1},\overline{(\cdot)}\overline{2}\cdots\overline{p-1}} \\ Q_{(\cdot)2\ldots p-1,\overline{1}\overline{(\cdot)}\cdots\overline{p-1}} & Q_{1(\cdot)\cdots p-1,\overline{1}\overline{(\cdot)}\cdots\overline{p-1}} & \cdots & Q_{12\cdots(\cdot)_{p-1},\overline{1}\overline{(\cdot)}\cdots\overline{p-1}} \\ \vdots & \vdots & \ddots & \vdots \\ Q_{(\cdot)2\cdots p-1,\overline{12}\cdots\overline{(\cdot)}_{p-1}} & Q_{1(\cdot)\cdots p-1,\overline{12}\cdots\overline{(\cdot)}_{p-1}} & \cdots & Q_{12\cdots(\cdot)_{p-1},\overline{12}\cdots\overline{(\cdot)}_{p-1}} \end{pmatrix}$$

and S satisfies that for vectors $X_1, \ldots, X_{p-1}, X$

$$(\overline{X}_1^{\mathsf{T}}, \ldots, \overline{X}_{p-1}^{\mathsf{T}}) \cdot S \cdot X = \sum_{\nu=1}^{p-1} \operatorname{div}''_X(\phi)_{v_1\cdots v_{p-1},\overline{v}_1\cdots\overline{X}_\nu\cdots\overline{v}_{p-1}} + \phi_{Xv_1\cdots v_{p-1},\overline{V}\overline{v}_1\cdots\overline{X}_\nu\cdots\overline{v}_{p-1}}.$$

To check that $(\frac{\partial}{\partial t} - \Delta) Q_{1\cdots p-1,\overline{1}\cdots\overline{p-1}} \geq 0$ we may extend V such that the following holds:

$$\nabla_i V = \frac{1}{t}\frac{\partial}{\partial z^i}, \quad \nabla_{\overline{i}} V = 0, \quad \left(\frac{\partial}{\partial t} - \Delta\right) V = -\frac{1}{t} V.$$

By using this set of equations, (4.18), (4.19), and (4.20) can be simplified to

$$\begin{aligned} &\left(\frac{\partial}{\partial t} - \Delta\right)(\operatorname{div}''_V(\phi))_{I_{p-1},\overline{J}_{p-1}} \\ &\quad = -\frac{1}{t}\operatorname{div}''_V(\phi) + \mathcal{KB}(\operatorname{div}''_V(\phi)) \\ &\qquad + \sum_{\nu=1}^{p-1} R^{k\overline{l}}_{V\,\overline{j}_\nu}(\operatorname{div}''(\phi))_{ki_1\cdots i_{p-1},\overline{j}_1\cdots(\overline{l})_\nu\cdots\overline{j}_{p-1}} \\ &\qquad - \frac{1}{2}\operatorname{div}''_{\operatorname{Ric}(V)}(\phi) - \frac{1}{t}\operatorname{div}'(\operatorname{div}''(\phi)), \end{aligned} \tag{5.3}$$

$$\begin{aligned} &\left(\frac{\partial}{\partial t} - \Delta\right)(\operatorname{div}'_{\overline{V}}(\phi))_{I_{p-1},\overline{J}_{p-1}} \\ &\quad = -\frac{1}{t}\operatorname{div}'_{\overline{V}}(\phi) + \mathcal{KB}(\operatorname{div}'_{\overline{V}}(\phi)) \\ &\qquad + \sum_{\mu=1}^{p-1} R^{k\overline{l}}_{i_\mu\,\overline{V}}(\operatorname{div}'(\phi))_{i_1\cdots(k)_\mu\cdots i_{p-1},\overline{l}\overline{j}_1\cdots\overline{j}_{p-1}} \\ &\qquad - \frac{1}{2}\operatorname{div}'_{\overline{\operatorname{Ric}(V)}}(\phi) - \frac{1}{t}\operatorname{div}''(\operatorname{div}'(\phi)), \end{aligned} \tag{5.4}$$

$$
\begin{aligned}
&\left(\frac{\partial}{\partial t}-\Delta\right)\phi_{V,\overline{V}}\\
&\quad=\mathcal{KB}(\phi_{V,\overline{V}})+R^{k\overline{l}}_{V\,\overline{V}}\phi_{k,\overline{l}}+\sum_{\nu=1}^{p-1}R^{k\overline{l}}_{V\,\overline{j}_\nu}\phi_{kI_{p-1},\overline{V}\,\overline{j}_1\cdots(\overline{l})_\nu\cdots\overline{j}_{p-1}}\\
&\quad\quad+\sum_{\mu=1}^{p-1}R^{k\overline{l}}_{i_\mu\,\overline{V}}\phi_{Vi_1\cdots(k)_\mu\cdots i_{p-1},\overline{l}\,\overline{J}_{p-1}}-\frac{1}{2}(\phi_{V,\overline{\mathrm{Ric}(V)}}+\phi_{\mathrm{Ric}(V),\overline{V}})\\
&\quad\quad-\frac{2}{t}\phi_{V,\overline{V}}-\frac{\Lambda\phi}{t^2}-\frac{1}{t}\,\mathrm{div}'_{\overline{V}}(\phi)-\frac{1}{t}\,\mathrm{div}''_{V}(\phi).
\end{aligned}
\tag{5.5}
$$

Adding (5.3), (5.4), and (5.5) with

$$
\left(\frac{\partial}{\partial t}-\Delta\right)\big[\mathrm{div}''(\mathrm{div}'(\phi))+\mathrm{div}'(\mathrm{div}''(\phi))\big]=\mathcal{KB}\big(\mathrm{div}''(\mathrm{div}'(\phi))+\mathrm{div}'(\mathrm{div}''(\phi))\big)
$$

and $(\frac{\partial}{\partial t}-\Delta)\Lambda\phi=\mathcal{KB}(\Lambda\phi)$ and using (5.1), we have that

$$
\begin{aligned}
&\left(\frac{\partial}{\partial t}-\Delta\right)Q_{I_{p-1},\overline{J}_{p-1}}\\
&\quad=\sum_{\nu=1}^{p-1}R^{k\overline{l}}_{V\,\overline{j}_\nu}\big(\phi_{kI_{p-1},\overline{V}\,\overline{j}_1\cdots(\overline{l})_\nu\cdots\overline{j}_{p-1}}+(\mathrm{div}''(\phi))_{kI_{p-1},\overline{j}_1\cdots(\overline{l})_\nu\cdots\overline{j}_{p-1}}\big)\\
&\quad\quad+\sum_{\mu=1}^{p-1}R^{k\overline{l}}_{i_\mu\,\overline{V}}\big(\phi_{Vi_1\cdots(k)_\mu\cdots i_{p-1},\overline{l}\,\overline{J}_{p-1}}+(\mathrm{div}'(\phi))_{i_1\cdots(k)_\mu\cdots i_{p-1},\overline{l}\,\overline{J}_{p-1}}\big)\\
&\quad\quad+R^{k\overline{l}}_{V\,\overline{V}}\phi_{kI_{p-1},\overline{l}\,\overline{J}_{p-1}}+\mathcal{KB}(Q)_{I_{p-1},\overline{J}_{p-1}}-\frac{2Q_{I_{p-1},\overline{J}_{p-1}}}{t}.
\end{aligned}
$$

Now the nonnegativity of $(\frac{\partial}{\partial t}-\Delta)Q_{1\cdots(p-1),\overline{1}\cdots(\overline{p-1})}$ at (x_0,t_0) can be proved in a similar way to the argument in Section 2. Observe that in $\mathcal{KB}(Q)_{1\cdots(p-1),\overline{1}\cdots(\overline{p-1})}$ the part involving only Ric is

$$
-\frac{1}{2}\sum_{i=1}^{p-1}(Q_{1\cdots\mathrm{Ric}(i)\cdots(p-1),\overline{1}\cdots(\overline{p-1})}+Q_{1\cdots(p-1),\overline{1}\cdots\overline{\mathrm{Ric}(i)}\cdots(\overline{p-1})})
$$

which vanishes due to (5.2). Hence we only need to establish the nonnegativity of

$$
\begin{aligned}
J\doteq&\sum_{\mu=1}^{p-1}\sum_{\nu=1}^{p-1}R^{k\overline{l}}_{\mu\,\overline{\nu}}Q_{1\cdots(k)_\mu\cdots(p-1),\overline{1}\cdots(\overline{l})_\nu\cdots\overline{p-1}}\\
&+\sum_{\nu=1}^{p-1}R^{k\overline{l}}_{V\,\overline{\nu}}\big(\phi_{k1\cdots(p-1),\overline{V}\,\overline{1}\cdots(\overline{l})_\nu\cdots(\overline{p-1})}+(\mathrm{div}''(\phi))_{k1\cdots(p-1),\overline{1}\cdots(\overline{l})_\nu\cdots(\overline{p-1})}\big)+
\end{aligned}
$$

$$+\sum_{\mu=1}^{p-1} R_{\mu}^{k\bar{l}}{}_{V\bar{\nu}}\big(\phi_{V1\cdots(k)_\mu\cdots(p-1),\bar{l}\bar{1}\cdots(\overline{p-1})} + (\mathrm{div}'(\phi))_{1\cdots(k)_\mu\cdots(p-1),\bar{l}\bar{1}\cdots(\overline{p-1})}\big)$$
$$+ R_{V\bar{V}}^{k\bar{l}}\phi_{k1\cdots(p-1),\bar{l}\bar{1}\cdots(\overline{p-1})}.$$

The curvature operator is in $\mathcal{C}_p$ implies that the matrix

$$\mathcal{M}_2 = \begin{pmatrix} R_{1\bar{1}(\cdot)(\bar{\cdot})} & R_{1\bar{2}(\cdot)(\bar{\cdot})} & \cdots & R_{1\overline{p-1}(\cdot)(\bar{\cdot})} & R_{1\bar{V}(\cdot)(\bar{\cdot})} \\ R_{2\bar{1}(\cdot)(\bar{\cdot})} & R_{2\bar{2}(\cdot)(\bar{\cdot})} & \cdots & R_{2\overline{p-1}(\cdot)(\bar{\cdot})} & R_{2\bar{V}(\cdot)(\bar{\cdot})} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ R_{p-1\bar{1}(\cdot)(\bar{\cdot})} & R_{p-1\bar{2}(\cdot)(\bar{\cdot})} & \cdots & R_{p-1\overline{p-1}(\cdot)(\bar{\cdot})} & R_{p-1\bar{V}(\cdot)(\bar{\cdot})} \\ R_{V\bar{1}(\cdot)(\bar{\cdot})} & R_{V\bar{2}(\cdot)(\bar{\cdot})} & \cdots & R_{V\overline{p-1}(\cdot)(\bar{\cdot})} & R_{V\bar{V}(\cdot)(\bar{\cdot})} \end{pmatrix} \geq 0.$$

The nonnegativity of J follows from $\mathrm{trace}(\mathcal{M}_1 \cdot \mathcal{M}_2) \geq 0$.

We define transformations on $T'M$, $(\mathrm{div}''(\phi))^{\bar{\nu}}$, $(\mathrm{div}'(\phi))^{\mu}$, $\phi_{\bar{V}}^{\bar{\nu}}$, and ϕ_V^{μ} by

$$\langle(\mathrm{div}''(\phi))^{\bar{\nu}}(X), \overline{Y}\rangle \doteqdot (\mathrm{div}''(\phi))_{X1\cdots(p-1),\bar{1}\cdots(\overline{Y})_\nu\cdots(\overline{p-1})},$$
$$\langle(\mathrm{div}'(\phi))^{\mu}(X), \overline{Y}\rangle \doteqdot (\mathrm{div}'(\phi))_{1\cdots(X)_\mu\cdots(p-1),\overline{Y1}\cdots(\overline{p-1})},$$
$$\langle\phi_{\bar{V}}^{\bar{\nu}}(X), \overline{Y}\rangle \doteqdot \phi_{X1\cdots(p-1),\overline{V1}\cdots(\overline{Y})_\nu\cdots(\overline{p-1})},$$
$$\langle\phi_V^{\mu}(X), \overline{Y}\rangle \doteqdot \phi_{V1\cdots(X)_\mu\cdots(p-1),\overline{Y1}\cdots(\overline{p-1})}.$$

Then the operator S defined previously can be written as

$$S = \oplus_{\nu=1}^{p-1}\big[(\mathrm{div}''(\phi))^{\bar{\nu}} + \phi_{\bar{V}}^{\bar{\nu}}\big].$$

If we define $Q^{\mu\bar{\nu}}$ in a similar way to $\phi^{\mu\bar{\nu}}$, then the quantity J above can be expressed as

$$J = \sum_{\mu,\nu=1}^{p-1} \mathrm{trace}(R^{\mu\bar{\nu}}Q^{\mu\bar{\nu}}) + \sum_{\nu=1}^{p-1} \mathrm{trace}\big(R^{V\bar{\nu}} \cdot ((\mathrm{div}''(\phi))^{\bar{\nu}} + \phi_{\bar{V}}^{\bar{\nu}})\big)$$
$$+ \sum_{\nu=1}^{p-1} \overline{\mathrm{trace}\big(R^{V\bar{\nu}} \cdot ((\mathrm{div}''(\phi))^{\bar{\nu}} + \phi_{\bar{V}}^{\bar{\nu}})\big)} + \mathrm{trace}(R^{V,\bar{V}} \cdot \phi^{p,\bar{p}}).$$

Hence one can modify the definitions of transformations $\mathcal{K}$ and $\mathcal{J}$ on $\bigoplus_{\mu=1}^{p} T'M$ in Section 2 so that $J = \mathrm{trace}(\mathcal{K} \cdot \mathcal{J})$, and $\mathcal{J}$ and $\mathcal{K}$ correspond to $\mathcal{M}_1$ and $\mathcal{M}_2$, respectively.

Remark 5.1. We suspect that the theorem (and later results) still holds even under the weaker assumption $\mathcal{C}_1$, even though the techniques employed here do not seem to be able to prove such a claim.

6 Hodge Laplacian Heat Equation Coupled with Kähler-Ricci Flow

Now we consider $(M^m, g(t))$, a complete solution of the Kähler-Ricci flow

$$\frac{\partial}{\partial t} g_{i\bar{j}} = -R_{i\bar{j}}. \tag{6.1}$$

Corollary 3.2 asserts that $\mathcal{C}_p$ is an invariant curvature condition under the Kähler-Ricci flow. Now we generalize the LYH estimate to the solution of (2.1). Again the result is proved for $p = 1$ and $p = m$ in [34] and [32], respectively.

THEOREM 6.1 *Let $(M, g(t))$ be a complete solution to the Kähler-Ricci flow* (6.1). *When M is noncompact, we assume that the curvature of (M, g) is uniformly bounded. Assume that ϕ is a positive solution to* (2.1). *Assume further that the curvature of $(M, g(t))$ satisfies $\mathcal{C}_p$. Then for any vector field V of $(1,0)$-type $\widetilde{Q} \geq 0$ for any $t > 0$, where*

$$\widetilde{Q} = Q + \mathrm{Ric}(\phi)$$

Here Q is the LYH quantity defined in Sections 4 *and* 5, *which is a $(p-1, p-1)$-form-valued (Hermitian) quadratic form of V, and* $\mathrm{Ric}(\phi)$ *is a $(p-1, p-1)$-form defined by*

$$\mathrm{Ric}(\phi)_{I_{p-1}, \overline{J}_{p-1}} \doteq g^{i\overline{l}} g^{k\bar{j}} R_{i\bar{j}} \phi_{kI_{p-1}, \overline{l}\overline{J}_{p-1}}.$$

Note that the difference between the above result and Theorem 4.1 is that the Laplacian operator $\Delta_{\overline{\partial}}$ is time dependent; namely, the $g_{i\bar{j}}$ and the connection used in the definition of $\overline{\partial}^*$ are evolved by the Kähler-Ricci flow equation. Moreover, since ∂^* and $\overline{\partial}^*$ depend on changing metrics now, the quantity Q is different from the static case even though they are defined by the same expression. Amazingly, the theorem asserts that the result still holds if we add a correction term $\mathrm{Ric}(\phi)$.

COROLLARY 6.2 *Let (M, g) and ϕ be as in Theorem* 6.1. *Assume that ϕ is d-closed and M is compact. Let $\psi = \Lambda\phi$. Then*

$$\frac{d}{dt}\left(t \int_M \psi \wedge \omega_0^{m-p+1} \right) \geq 0. \tag{6.2}$$

Here ω_0 is the Kähler form of the initial metric.

PROOF. Note that $\frac{\partial}{\partial t}\psi + \Delta_{\overline{\partial}}\psi = \mathrm{Ric}(\phi)$, and the operators ∂ and $\overline{\partial}$ can be commuted with $\frac{\partial}{\partial t}$ and $\Delta_{\overline{\partial}}$. The rest is the same as the proof of Corollary 4.2. □

We first start with some lemmas that are the time-dependent versions of Lemmas 4.5 and 4.6.

LEMMA 6.3 *Let ϕ be a (p, p)-form satisfying* (2.1)*. Then under a normal coordinate,*

$$\begin{aligned}
&\left(\frac{\partial}{\partial t}-\Delta\right)(\mathrm{div}''(\phi))_{iI_{p-1},\overline{J}_{p-1}}\\
&\quad=\mathcal{KB}(\mathrm{div}_i''(\phi))_{I_{p-1},\overline{J}_{p-1}}\\
&\quad\quad+\sum_{\nu=1}^{p-1}R_{i\,\bar{j}_\nu l\bar{k}}(\mathrm{div}''(\phi))_{kI_{p-1},\bar{j}_1\cdots(\bar{l})_\nu\cdots\bar{j}_{p-1}}-\frac{1}{2}R_{i\bar{k}}(\mathrm{div}''(\phi))_{kI_{p-1},\overline{J}_{p-1}}\\
&\quad\quad+R_{j\bar{k}}\nabla_k\phi_{iI_{p-1},\overline{jJ}_{p-1}}+\nabla_iR_{j\bar{k}}\phi_{kI_{p-1},\overline{jJ}_{p-1}}\\
&\quad\quad+\sum_{\mu=1}^{p-1}\nabla_{i_\mu}R_{l\bar{k}}\phi_{ii_1\cdots(k)_\mu\cdots i_{p-1},\overline{lJ}_{p-1}},
\end{aligned}\tag{6.3}$$

$$\begin{aligned}
&\left(\frac{\partial}{\partial t}-\Delta\right)(\mathrm{div}'(\phi))_{I_{p-1},\overline{jJ}_{p-1}}\\
&\quad=\mathcal{KB}(\mathrm{div}'_{\bar{j}}(\phi))_{I_{p-1},\overline{J}_{p-1}}+\sum_{\mu=1}^{p-1}R_{i_\mu\bar{j}\,l\bar{k}}(\mathrm{div}'(\phi))_{i_1\cdots(k)_\mu\cdots i_{p-1},\overline{lJ}_{p-1}}\\
&\quad\quad-\frac{1}{2}R_{l\bar{j}}(\mathrm{div}'(\phi))_{I_{p-1},\overline{lJ}_{p-1}}+R_{l\bar{k}}\nabla_{\bar{l}}\phi_{kI_{p-1},\overline{jJ}_{p-1}}\\
&\quad\quad+\nabla_{\bar{j}}R_{l\bar{k}}\phi_{kI_{p-1},\overline{lJ}_{p-1}}+\sum_{\nu=1}^{p-1}\nabla_{\bar{j}_\mu}R_{l\bar{k}}\phi_{kI_{p-1},\bar{j}\bar{j}_1\cdots(\bar{l})_\nu\cdots\bar{j}_{p-1}}.
\end{aligned}\tag{6.4}$$

PROOF. Since $\frac{\partial}{\partial t}\Gamma^h_{jl}=-g^{h\bar{q}}\nabla_jR_{l\bar{q}}$,

$$\begin{aligned}
&\frac{\partial}{\partial t}\Big(\sum g^{\bar{l}k}\nabla_k\phi_{ii_1\cdots i_{p-1},\bar{l}\bar{j}_1\cdots\bar{j}_{p-1}}\Big)\\
&\quad=R_{l\bar{k}}\nabla_k\phi_{iI_{p-1},\overline{lJ}_{p-1}}+\nabla_iR_{l\bar{k}}\phi_{kI_{p-1},\overline{lJ}_{p-1}}\\
&\quad\quad+\sum_{\mu=1}^{p-1}\nabla_{i_\mu}R_{l\bar{k}}\phi_{ii_1\cdots(k)_\mu\cdots i_{p-1},\overline{lJ}_{p-1}}+\left(\mathrm{div}''\left(\frac{\partial}{\partial t}\phi\right)\right)_{iI_{p-1},\overline{J}_{p-1}},
\end{aligned}\tag{6.5}$$

the first equation follows from the fact that $\Delta_{\bar{\partial}}$ is commutative with div'' and (4.15). The second evolution equation follows from taking the conjugation of the first one. □

LEMMA 6.4 *Let ϕ be a solution to* (2.1)*. Then under the normal coordinate*

$$\begin{aligned}
&\left(\frac{\partial}{\partial t}-\Delta\right)\big(\mathrm{div}'(\mathrm{div}''(\phi))\big)_{I_{p-1},\overline{J}_{p-1}}\\
&\quad=\left(\frac{\partial}{\partial t}-\Delta\right)\big(\mathrm{div}''(\mathrm{div}'(\phi))\big)_{I_{p-1},\overline{J}_{p-1}}=
\end{aligned}$$

$$= \big(\mathcal{KB}(\mathrm{div}'(\mathrm{div}''(\phi)))\big)_{I_{p-1},\overline{J}_{p-1}} + \mathcal{E}(\phi)_{I_{p-1},\overline{J}_{p-1}}$$
$$= \big(\mathcal{KB}(\mathrm{div}''(\mathrm{div}'(\phi)))\big)_{I_{p-1},\overline{J}_{p-1}} + \mathcal{E}(\phi)_{I_{p-1},\overline{J}_{p-1}},$$

where

$$\begin{aligned}
&\mathcal{E}(\phi)_{I_{p-1},\overline{J}_{p-1}}\\
&\quad \doteq R_{j\bar{i}}\big(\nabla_i(\mathrm{div}'(\phi))_{I_{p-1},\overline{jJ}_{p-1}} + \nabla_{\bar{j}}(\mathrm{div}''(\phi))_{iI_{p-1},\overline{J}_{p-1}}\big)\\
&\qquad + \sum_{\mu=1}^{p-1} \nabla_j R_{i_\mu\bar{k}}(\mathrm{div}'(\phi))_{i_1\cdots(k)_\mu\cdots i_{p-1},\overline{jJ}_{p-1}}\\
&\qquad + \sum_{\nu=1}^{p-1} \nabla_{\bar{i}} R_{l\bar{j}_\nu}(\mathrm{div}''(\phi))_{iI_{p-1},\bar{j}_1\cdots(\bar{l})_\nu\cdots\bar{j}_{p-1}}\\
&\qquad + \sum_{\mu=1}^{p-1} R_{l\bar{k}}R_{j\bar{i}i_\mu\bar{l}}\phi_{ii_1\cdots(k)_\mu\cdots i_{p-1},\overline{jJ}_{p-1}}\\
&\qquad + \sum_{\nu=1}^{p-1} R_{l\bar{k}}R_{j\bar{i}k\bar{j}_\nu}\phi_{iI_{p-1},\bar{j}\bar{j}_1\cdots(\bar{l})_\nu\cdots\bar{j}_{p-1}}\\
&\qquad + \Delta R_{j\bar{i}}\phi_{iI_{p-1},\overline{jJ}_{p-1}} + \nabla_k R_{j\bar{i}}\nabla_{\bar{k}}\phi_{iI_{p-1},\overline{jJ}_{p-1}} + \nabla_{\bar{k}}R_{j\bar{i}}\nabla_k\phi_{iI_{p-1},\overline{jJ}_{p-1}}\\
&\qquad + R_{j\bar{k}}R_{k\bar{i}}\phi_{iI_{p-1},\overline{jJ}_{p-1}} - R_{k\bar{l}}R_{l\bar{k}j\bar{i}}\phi_{iI_{p-1},\overline{jJ}_{p-1}}.
\end{aligned}$$

PROOF. First note that $\mathrm{div}'(\mathrm{div}''(\phi)) = \mathrm{div}''(\mathrm{div}''(\phi))$, hence the first equation. Compute

$$\begin{aligned}
&\frac{\partial}{\partial t}\big(g^{i\bar{j}}\nabla_{\bar{j}}(\mathrm{div}''(\phi))_{iI_{p-1},\overline{J}_{p-1}}\big)\\
&\quad = R_{j\bar{i}}\nabla_{\bar{j}}(\mathrm{div}''(\phi))_{iI_{p-1},\overline{J}_{p-1}} + \sum_{\nu=1}^{p-1}\nabla_{\bar{i}}R_{l\bar{j}_\nu}(\mathrm{div}''(\phi))_{iI_{p-1},\bar{j}_1\cdots(\bar{l})_\nu\cdots\bar{j}_{p-1}}\\
&\qquad + \nabla_{\bar{i}}\bigg(\frac{\partial}{\partial t}\big((\mathrm{div}''(\phi))_{iI_{p-1},\overline{J}_{p-1}}\big)\bigg).
\end{aligned}$$

Now we plug in (6.5). Applying the commutator formula, the second Bianchi identity, and Lemma 4.6, we get the second equation. The third one follows from the second by taking the conjugation. □

The next lemma is on $\mathrm{Ric}(\phi)$. The proof is via straightforward computation.

LEMMA 6.5 *For ϕ, a solution of* (2.1)*, under a normal coordinate,*

$$
\begin{aligned}
&\left(\frac{\partial}{\partial t}-\Delta\right)(R_{j\bar{i}}\phi_{iI_{p-1},\overline{jJ}_{p-1}})\\
&\quad=\sum_{\mu=1}^{p-1}\sum_{\nu=1}^{p-1}R^{k\bar{l}}_{i_\mu\ \bar{j}_\nu}R_{j\bar{i}}\phi_{ii_1\cdots(k)_\mu\cdots i_{p-1},\bar{j}\bar{j}_1\cdots(\bar{l})_\nu\cdots\bar{j}_{p-1}}\\
&\qquad-\frac{1}{2}\Bigg(\sum_{\nu=1}^{p-1}R^{\bar{l}}_{\bar{j}_\nu}R_{j\bar{i}}\phi_{iI_{p-1},\bar{j}\bar{j}_1\cdots(\bar{l})_\nu\cdots\bar{j}_{p-1}}\\
&\qquad\qquad+\sum_{\mu=1}^{p-1}R^{k}_{i_\mu}R_{j\bar{i}}\phi_{ii_1\cdots(k)_\mu\cdots i_{p-1},\overline{jJ}_{p-1}}\Bigg)\\
&\qquad-\Bigg(\sum_{\mu=1}^{p-1}R_{l\bar{k}}R_{j\bar{i}i_\mu\bar{l}}\phi_{ii_1\cdots(k)_\mu\cdots i_{p-1},\overline{jJ}_{p-1}}\\
&\qquad\qquad+\sum_{\nu=1}^{p-1}R_{l\bar{k}}R_{j\bar{i}k\bar{j}_\nu}\phi_{iI_{p-1},\bar{j}\bar{j}_1\cdots(\bar{l})_\nu\cdots\bar{j}_{p-1}}\Bigg)\\
&\qquad-\left(\nabla_kR_{j\bar{i}}\nabla_{\bar{k}}\phi_{iI_{p-1},\overline{jJ}_{p-1}}+\nabla_{\bar{k}}R_{j\bar{i}}\nabla_k\phi_{iI_{p-1},\overline{jJ}_{p-1}}\right)\\
&\qquad+2R_{k\bar{l}}R_{l\bar{k}j\bar{i}}\phi_{iI_{p-1},\overline{jJ}_{p-1}}.
\end{aligned}
\tag{6.6}
$$

Adapting the notation from Section 4, Lemma 6.3 implies the following set of formulae:

$$
\begin{aligned}
&\left(\frac{\partial}{\partial t}-\Delta\right)(\mathrm{div}''_V(\phi))_{I_{p-1},\bar{J}_{p-1}}\\
&\quad=\mathrm{div}''_{(\frac{\partial}{\partial t}-\Delta)V}(\phi)+\mathcal{KB}(\mathrm{div}''_V(\phi))\\
&\qquad+\sum_{\nu=1}^{p-1}R^{k\bar{l}}_{V\ \bar{j}_\nu}(\mathrm{div}''(\phi))_{ki_1\cdots i_{p-1},\bar{j}_1\cdots(\bar{l})_\nu\cdots\bar{j}_{p-1}}-\frac{1}{2}\,\mathrm{div}''_{\mathrm{Ric}(V)}(\phi)\\
&\qquad+R_{j\bar{k}}\nabla_k\phi_{Vi_1\cdots i_{p-1},\bar{j}\bar{j}_1\cdots\bar{j}_{p-1}}+\nabla_VR_{j\bar{i}}\phi_{ii_1\cdots i_{p-1},\bar{j}\bar{j}_1\cdots\bar{j}_{p-1}}\\
&\qquad+\sum_{\mu=1}^{p-1}\nabla_{i_\mu}R_{j\bar{k}}\phi_{Vi_1\cdots(k)_\mu\cdots i_{p-1},\bar{j}\bar{j}_1\cdots\bar{j}_{p-1}}\\
&\qquad-g^{i\bar{j}}\big((\nabla_i\,\mathrm{div}''(\phi))_{\nabla_{\bar{j}}V}+(\nabla_{\bar{j}}\,\mathrm{div}''(\phi))_{\nabla_iV}\big),
\end{aligned}
\tag{6.7}
$$

$$
\begin{aligned}
&\left(\frac{\partial}{\partial t}-\Delta\right)(\operatorname{div}'_{\overline{V}}(\phi))_{I_{p-1},\bar{J}_{p-1}}\\
&\quad=\operatorname{div}'_{(\frac{\partial}{\partial t}-\Delta)\overline{V}}(\phi)+\mathcal{KB}(\operatorname{div}'_{\overline{V}}(\phi))\\
&\qquad+\sum_{\mu=1}^{p-1}R^{k\bar{l}}_{i_\mu\,\overline{V}}(\operatorname{div}'(\phi))_{i_1\cdots(k)_\mu\cdots i_{p-1},\bar{l}\bar{j}_1\cdots\bar{j}_{p-1}}-\frac{1}{2}\operatorname{div}'_{\overline{\mathrm{Ric}(V)}}(\phi)\\
&\qquad+R_{k\bar{l}}\nabla_{\bar{k}}\phi_{ii_1\cdots i_{p-1},\overline{V}\bar{j}_1\cdots\bar{j}_{p-1}}+\nabla_{\overline{V}}R_{j\bar{i}}\phi_{ii_1\cdots i_{p-1},\bar{j}\bar{j}_1\cdots\bar{j}_{p-1}}\\
&\qquad+\sum_{\nu=1}^{p-1}\nabla_{\bar{j}_\nu}R_{j\bar{i}}\phi_{ii_1\cdots i_{p-1},\overline{V}\bar{j}_1\cdots(\bar{j})_\nu\cdots\bar{j}_{p-1}}\\
&\qquad-g^{i\bar{j}}\big((\nabla_i\operatorname{div}'(\phi))_{\nabla_{\bar{j}}\overline{V}}+(\nabla_{\bar{j}}\operatorname{div}'(\phi))_{\nabla_i\overline{V}}\big).
\end{aligned}
\tag{6.8}
$$

For $\Lambda\phi$, we have the following evolution equation:

$$
\left(\frac{\partial}{\partial t}-\Delta\right)(\Lambda\phi)_{i_1\cdots i_{p-1},\bar{j}_1\cdots\bar{j}_{p-1}}=\mathcal{KB}(\Lambda\phi)_{i_1\cdots i_{p-1},\bar{j}_1\cdots\bar{j}_{p-1}}+R_{j\bar{i}}\phi_{ii_1\cdots i_{p-1},\bar{j}\bar{j}_1\cdots\bar{j}_{p-1}}.
\tag{6.9}
$$

7 LYH Type Estimates for Kähler-Ricci Flow under Condition $\mathcal{C}_p$

Let $(M,g(t))$ be a complete solution to the Kähler-Ricci flow. Assume further that the curvature operator satisfies $\mathcal{C}_p$. Let

$$
\mathcal{M}_{\alpha\bar{\beta}}=\Delta R_{\alpha\bar{\beta}}+R_{\alpha\bar{\beta}\gamma\bar{\delta}}R_{\delta\bar{\gamma}}+\frac{R_{\alpha\bar{\beta}}}{t},\quad P_{\alpha\bar{\beta}\gamma}=\nabla_\gamma R_{\alpha\bar{\beta}},\quad P_{\alpha\bar{\beta}\bar{\gamma}}=\nabla_{\bar{\gamma}}R_{\alpha\bar{\beta}}.
$$

Also, let $P_{\bar{\beta}\alpha\gamma}=\nabla_\gamma R_{\bar{\beta}\alpha}$ and $P_{\bar{\beta}\alpha\bar{\gamma}}=\nabla_{\bar{\gamma}}R_{\bar{\beta}\alpha}$. Clearly $P_{\alpha\bar{\beta}\gamma}=P_{\bar{\beta}\alpha\gamma}$ and $P_{\alpha\bar{\beta}\bar{\gamma}}=P_{\bar{\beta}\alpha\bar{\gamma}}$. The second Bianchi identity implies that

$$
P_{\alpha\bar{\beta}\gamma}=P_{\gamma\bar{\beta}\alpha},\quad \overline{P_{\alpha\bar{\beta}\gamma}}=P_{\beta\bar{\alpha}\bar{\gamma}}=P_{\bar{\alpha}\beta\bar{\gamma}}.
$$

THEOREM 7.1 *Let $(M,g(t))$ be a complete solution to the Kähler-Ricci flow satisfying the condition $\mathcal{C}_p$ on $M\times[0,T]$. When M is noncompact, we assume that the curvature of $(M,g(t))$ is bounded on $M\times[0,T]$. Then for any $\bigwedge^{1,1}$-vector U that can be written as $U=\sum_{i=1}^{p-1}X_i\wedge\bar{Y}_i+W\wedge\bar{V}$, for $(1,0)$-type vectors X_i, Y_i, W, and V, the Hermitian bilinear form $\mathcal{Q}$ defined as*

$$
\begin{aligned}
\mathcal{Q}(U\oplus W)\doteq{}&\mathcal{M}_{\alpha\bar{\beta}}W^\alpha W^{\bar{\beta}}+P_{\alpha\bar{\beta}\gamma}\bar{U}^{\bar{\beta}\gamma}W^\alpha\\
&+P_{\alpha\bar{\beta}\bar{\gamma}}U^{\alpha\bar{\gamma}}W^{\bar{\beta}}+R_{\alpha\bar{\beta}\gamma\bar{\delta}}U^{\alpha\bar{\beta}}\bar{U}^{\bar{\delta}\gamma}
\end{aligned}
\tag{7.1}
$$

satisfies that $\mathcal{Q}\geq 0$ for any $t>0$. Moreover, if the equality ever occurs for some $t>0$, the universal cover of $(M,g(t))$ must be a gradient-expanding Kähler-Ricci soliton.

The theorem says that for any vector W, viewing $\mathcal{Q}$ as a Hermitian bilinear form on $\bigwedge^{1,1}$ space, it also satisfies $\mathcal{C}_p$, but only for the $\bigwedge^{1,1}$ vector U with the form $U = \sum_{i=1}^{p} X_i \wedge \overline{Y}_i$ with $X_p = W$. If we define $P : T'M \to \bigwedge^{1,1}$ by the equation $\langle P(W), \overline{U}\rangle = P_{\alpha\bar{\beta}\gamma}\overline{U}^{\bar{\beta}\gamma}W^{\alpha}$, the LYH expression can be written, by abusing the notation with $\mathcal{Q}$ also denoting the Hermitian symmetric transformation, as

$$\langle \mathcal{Q}(U), \overline{U}\rangle = \langle \mathcal{M}(W), \overline{W}\rangle + 2\,\mathrm{Re}(\langle P(W), \overline{U}\rangle) + \langle \mathrm{Rm}(U), \overline{U}\rangle. \tag{7.2}$$

Remark 7.2. When $p = 1$, inequality (7.1) recovers the LYH inequality of Cao [8]. When $p > 1$, $\mathcal{Q}$ can be written as

$$\begin{aligned}\mathcal{Q} = {} & Z_{\alpha\bar{\beta}}W^{\alpha}W^{\bar{\beta}} + (P_{\alpha\bar{\beta}\gamma} + R_{\alpha\bar{\nu}\gamma\bar{\beta}})\overline{\tilde{U}}^{\bar{\beta}\gamma}W^{\alpha} \\ & + (P_{\alpha\bar{\beta}\bar{\gamma}} + R_{\nu\bar{\beta}\alpha\bar{\gamma}})\tilde{U}^{\alpha\bar{\gamma}}W^{\bar{\beta}} + R_{\alpha\bar{\beta}\gamma\bar{\delta}}\tilde{U}^{\alpha\bar{\beta}}\overline{\tilde{U}}^{\bar{\delta}\gamma}\end{aligned}$$

with $\tilde{U} = \sum_{i=1}^{p-1} X_i \wedge \overline{Y}_i$ and

$$Z_{\alpha\bar{\beta}} \doteqdot \mathcal{M}_{\alpha\bar{\beta}} + P_{\alpha\bar{\beta}\gamma}V^{\gamma} + P_{\alpha\bar{\beta}\bar{\gamma}}V^{\bar{\gamma}} + R_{\alpha\bar{\beta}\gamma\bar{\delta}}V^{\gamma}V^{\bar{\delta}}. \tag{7.3}$$

Equivalently, if we write the above as $\langle Z(W \wedge \overline{V}), \overline{W \wedge \overline{V}}\rangle$,

$$\mathcal{Q} = \langle Z(W \wedge \overline{V}), \overline{W \wedge \overline{V}}\rangle + 2\,\mathrm{Re}(\widetilde{P}(W \wedge \overline{V}), \overline{\tilde{U}}) + \langle \mathrm{Rm}(\tilde{U}), \overline{\tilde{U}}\rangle.$$

Here $\widetilde{P}$ is defined as $\widetilde{P}(W \wedge \overline{V}) = P(W) + \mathrm{Rm}(W \wedge \overline{V})$.

Note that Hamilton in [17] proved that under the stronger assumption that the curvature operator $\mathrm{Rm} \geq 0$, $Q(U \oplus W) \geq 0$ for any $\bigwedge^2$-vector U. For p sufficiently large, $\mathcal{C}_p$ is equivalent to $\mathrm{Rm} \geq 0$ and by taking $U = U_1 + W \wedge \overline{Y}_p$ with $U_1 = U - W \wedge \overline{Y}_p$, one can see that the above result implies Hamilton's result. Hence Theorem 7.1 interpolates between Cao's result and Hamilton's result for the Kähler-Ricci flow. In a later section we shall prove another set of estimates that generalize Hamilton's estimate for the Ricci flow on Riemannian manifolds.

One can easily get the following lemma through direct calculation, which can also be derived from lemmas 4.3 and 4.4 of [17]:

LEMMA 7.3 *Let $(M, g(t))$ be a solution to the Kähler-Ricci flow. Then we have*:

$$\begin{aligned}&\left(\frac{\partial}{\partial t} - \Delta\right)\mathcal{M}_{\alpha\bar{\beta}} \\ &= R_{\alpha\bar{\beta}\gamma\bar{\delta}}\mathcal{M}_{\delta\bar{\gamma}} - \frac{1}{2}(R_{\alpha\bar{\eta}}\mathcal{M}_{\eta\bar{\beta}} + R_{\eta\bar{\beta}}\mathcal{M}_{\alpha\bar{\eta}}) + R_{\alpha\bar{\beta}\gamma\bar{\delta}}R_{\delta\bar{\xi}}R_{\xi\bar{\gamma}} \\ &\quad + R_{\delta\bar{\gamma}}(\nabla_{\gamma}P_{\alpha\bar{\beta}\bar{\delta}} + \nabla_{\bar{\delta}}P_{\alpha\bar{\beta}\gamma}) + P_{\alpha\bar{\xi}\gamma}P_{\xi\bar{\beta}\bar{\gamma}} - P_{\alpha\bar{\xi}\bar{\gamma}}P_{\xi\bar{\beta}\gamma} - \frac{R_{\alpha\bar{\beta}}}{t^2},\end{aligned} \tag{7.4}$$

$$
\begin{aligned}
&\left(\frac{\partial}{\partial t}-\Delta\right)P_{\alpha\bar{\beta}\gamma}\\
&\quad=R_{\alpha\bar{\beta}\xi\bar{\eta}}P_{\eta\bar{\xi}\gamma}+R_{\xi\bar{\beta}\gamma\bar{\delta}}P_{\alpha\bar{\xi}\delta}-R_{\alpha\bar{\xi}\gamma\bar{\delta}}P_{\xi\bar{\beta}\delta}\\
&\qquad-\frac{1}{2}(R_{\alpha\bar{\xi}}P_{\xi\bar{\beta}\gamma}+R_{\xi\bar{\beta}}P_{\alpha\bar{\xi}\gamma}+R_{\gamma\bar{\xi}}P_{\alpha\bar{\beta}\xi})+\nabla_{\gamma}R_{\alpha\bar{\beta}\xi\bar{\eta}}R_{\eta\bar{\xi}}.
\end{aligned}
\tag{7.5}
$$

By taking the conjugation of (7.5), we have

$$
\begin{aligned}
&\left(\frac{\partial}{\partial t}-\Delta\right)P_{\alpha\bar{\beta}\bar{\gamma}}\\
&\quad=R_{\alpha\bar{\beta}\xi\bar{\eta}}P_{\eta\bar{\xi}\bar{\gamma}}+R_{\alpha\bar{\xi}\delta\bar{\gamma}}P_{\xi\bar{\beta}\bar{\delta}}-R_{\xi\bar{\beta}\delta\bar{\gamma}}P_{\alpha\bar{\xi}\bar{\delta}}\\
&\qquad-\frac{1}{2}(R_{\alpha\bar{\xi}}P_{\xi\bar{\beta}\bar{\gamma}}+R_{\xi\bar{\beta}}P_{\alpha\bar{\xi}\bar{\gamma}}+R_{\xi\bar{\gamma}}P_{\alpha\bar{\beta}\bar{\xi}})+\nabla_{\bar{\gamma}}R_{\alpha\bar{\beta}\xi\bar{\eta}}R_{\eta\bar{\xi}}.
\end{aligned}
\tag{7.6}
$$

The evolution equation for the curvature tensor is (see, for example, [3])

$$
\begin{aligned}
&\left(\frac{\partial}{\partial t}-\Delta\right)R_{\alpha\bar{\beta}\gamma\bar{\delta}}\\
&\quad=R_{\alpha\bar{\beta}\xi\bar{\eta}}R_{\eta\bar{\xi}\gamma\bar{\delta}}-R_{\alpha\bar{\xi}\gamma\bar{\eta}}R_{\xi\bar{\beta}\eta\bar{\delta}}+R_{\alpha\bar{\delta}\xi\bar{\eta}}R_{\eta\bar{\xi}\gamma\bar{\beta}}\\
&\qquad-\frac{1}{2}(R_{\alpha\bar{\xi}}R_{\xi\bar{\beta}\gamma\bar{\delta}}+R_{\xi\bar{\beta}}R_{\alpha\bar{\xi}\gamma\bar{\delta}}+R_{\gamma\bar{\xi}}R_{\alpha\bar{\beta}\xi\bar{\delta}}+R_{\xi\bar{\delta}}R_{\alpha\bar{\beta}\gamma\bar{\xi}}).
\end{aligned}
\tag{7.7}
$$

Now we begin to prove the theorem. We assume that the curvature of $(M, g(t))$ satisfies $\mathcal{C}_p$. (We also say that Rm has p-semipositive bisectional curvature.) One can adapt the perturbation argument as [17] if Rm does not have strictly p-positive bisectional curvature. Hence when the manifold is compact, without loss of generality we may assume that Rm has strictly p-positive bisectional curvature.

Then it is clear that when t is small, $\mathcal{Q}$ is positive, since the bisectional curvature is strictly p-positive and $\mathcal{M}_{\alpha\bar{\beta}}$ has a term $R_{\alpha\bar{\beta}}/t$. We claim $\mathcal{Q}\geq 0$ for all time. If it fails to hold, there is a first time t_0, a point x_0, and $U\in\Lambda^{1,1}T_{x_0}M$, $W\in\Lambda^{1,0}$, such that $\mathcal{Q}(U\oplus W)=0$, and for any $t\leq t_0$, $x\in M$, $(1,1)$-vector $\hat{U}\in\Lambda^{1,1}T_xM$ and $(1,0)$-vector $\hat{W}\in T_xM$, $\mathcal{Q}(\hat{U}\oplus\hat{W})\geq 0$. We extend U and W in space-time at (x_0,t_0) in the following way:

$$
\begin{aligned}
\left(\frac{\partial}{\partial t}-\Delta\right)U^{\gamma\bar{\delta}}&=\frac{1}{2}\big(R^{\gamma}_{\alpha}U^{\alpha\bar{\delta}}+R^{\bar{\delta}}_{\bar{\alpha}}U^{\gamma\bar{\alpha}}\big),\\
\left(\frac{\partial}{\partial t}-\Delta\right)W^{\alpha}&=\frac{1}{2}R^{\alpha}_{\beta}W^{\beta}+\frac{1}{t}W^{\alpha},\\
\nabla_{\bar{s}}U^{\gamma\bar{\delta}}=R^{\bar{\delta}}_{\bar{s}}W^{\gamma}+\frac{1}{t}g^{\bar{\delta}}_{\bar{s}}W^{\gamma},&\quad \nabla_{s}U^{\gamma\bar{\delta}}=0,\quad \nabla_{\gamma}W^{\alpha}=\nabla_{\bar{\gamma}}W^{\alpha}=0.
\end{aligned}
$$

Here R^{α}_{β} and $R^{\bar{\gamma}}_{\bar{\delta}}$ are the associated tensors obtained by raising the indices on the Ricci tensor. These sets of equations are the same as those of [17] in disguise. As in [17], it suffices to check that at the point (x_0,t_0), $(\frac{\partial}{\partial t}-\Delta)\mathcal{Q}\geq 0$. Using the above

equations and equations (7.4), (7.5), (7.6), and (7.7), a lengthy but straightforward computation shows that

$$\begin{aligned}\left(\frac{\partial}{\partial t}-\Delta\right)\mathcal{Q} &= R_{\alpha\bar\beta\gamma\bar\delta}M_{\delta\bar\gamma}W^{\alpha}W^{\bar\beta}+R_{\alpha\bar\beta\xi\bar\eta}P_{\eta\bar\xi\gamma}\bar U^{\bar\beta\gamma}W^{\alpha}\\ &\quad+R_{\alpha\bar\beta\xi\bar\eta}P_{\eta\bar\xi\bar\gamma}U^{\alpha\bar\gamma}W^{\bar\beta}+R_{\alpha\bar\delta\xi\bar\eta}R_{\eta\bar\xi\gamma\bar\beta}U^{\alpha\bar\beta}\bar U^{\bar\delta\gamma}\\ &\quad-\big(P_{\alpha\bar\xi\bar\gamma}P_{\xi\bar\beta\gamma}W^{\alpha}W^{\bar\beta}+R_{\alpha\bar\xi\gamma\bar\delta}P_{\xi\bar\beta\delta}\bar U^{\bar\beta\gamma}W^{\alpha}.\\ &\qquad+R_{\xi\bar\beta\delta\bar\gamma}P_{\alpha\overline{\xi\delta}}U^{\alpha\bar\gamma}W^{\bar\beta}+R_{\alpha\bar\xi\gamma\bar\eta}R_{\xi\bar\beta\eta\bar\delta}U^{\alpha\bar\beta}\bar U^{\bar\delta\gamma}\big)\\ &\quad+\big(P_{\alpha\bar\xi\gamma}W^{\alpha}+R_{\alpha\bar\xi\gamma\bar\delta}U^{\alpha\bar\delta}\big)\big(P_{\xi\bar\beta\bar\gamma}W^{\bar\beta}+R_{\xi\bar\beta\delta\bar\gamma}\bar U^{\bar\beta\delta}\big).\end{aligned}$$

The above computation can also be derived using lemma 4.5 of [17].

In the following, $X_p = W, Y_p = V$. To prove $(\frac{\partial}{\partial t}-\Delta)\mathcal{Q}\geq 0$ it is enough to show the nonnegativity of

$$\begin{aligned}\mathcal{J} &\doteqdot R_{X_p\bar X_p\gamma\bar\delta}Z_{\delta\bar\gamma}+\sum_{\nu=1}^{p-1}R_{X_p\bar X_\nu\gamma\bar\delta}(P_{\delta\bar\gamma Y_\nu}+R_{\delta\bar\gamma Y_\nu\bar Y_p})\\ &\quad+\sum_{\mu=1}^{p-1}R_{X_\mu\bar X_p\gamma\bar\delta}(P_{\delta\bar\gamma\bar Y_\mu}+R_{\delta\bar\gamma Y_p\bar Y_\mu})\\ &\quad+\sum_{\mu,\nu=1}^{p-1}R_{X_\mu\bar X_\nu\gamma\bar\delta}R_{\delta\bar\gamma Y_\nu\bar Y_\mu}+\left|P_{X_p\bar\beta\alpha}+\sum_{\mu=1}^{p}R_{X_\mu\bar Y_\mu\alpha\bar\beta}\right|^2\\ &\quad-\left(|P_{X_p\bar\gamma\bar\delta}|^2+\sum_{\nu=1}^{p}R_{X_p\bar\gamma Y_\nu\bar\delta}P_{\gamma\bar X_\nu\delta}\right.\\ &\qquad\left.+\sum_{\mu=1}^{p}R_{\gamma\bar X_p\delta\bar Y_\mu}P_{X_\mu\bar\gamma\bar\delta}+\sum_{\mu,\nu=1}^{p}R_{X_\mu\bar\gamma Y_\nu\bar\delta}R_{\gamma\bar Y_\mu\delta\bar X_\nu}\right),\end{aligned}$$

where we have replaced U and W, respectively, by $\sum_{i=1}^{p}X_i\wedge\bar Y_i$ and X_p.

Let

$$A_1=\begin{pmatrix}R_{X_1\bar X_1(\cdot)\bar(\cdot)} & R_{X_1\bar X_2(\cdot)\bar(\cdot)} & \cdots & R_{X_1\bar X_p(\cdot)\bar(\cdot)}\\ R_{X_2\bar X_1(\cdot)\bar(\cdot)} & R_{X_2\bar X_2(\cdot)\bar(\cdot)} & \cdots & R_{X_2\bar X_p(\cdot)\bar(\cdot)}\\ \vdots & \vdots & \ddots & \vdots\\ R_{X_p\bar X_1(\cdot)\bar(\cdot)} & R_{X_p\bar X_2(\cdot)\bar(\cdot)} & \cdots & R_{X_p\bar X_p(\cdot)\bar(\cdot)}\end{pmatrix},$$

$$A_2 = \begin{pmatrix} R_{Y_1\bar{Y}_1(\cdot)\bar{(\cdot)}} & R_{Y_1\bar{Y}_2(\cdot)\bar{(\cdot)}} & \cdots & R_{Y_1\bar{Y}_{p-1}(\cdot)\bar{(\cdot)}} & E^{1\bar{p}}_{(\cdot)\bar{(\cdot)}} \\ R_{Y_2\bar{Y}_1(\cdot)\bar{(\cdot)}} & R_{Y_2\bar{Y}_2(\cdot)\bar{(\cdot)}} & \cdots & R_{Y_2\bar{Y}_{p-1}(\cdot)\bar{(\cdot)}} & E^{2\bar{p}}_{(\cdot)\bar{(\cdot)}} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ R_{Y_{p-1}\bar{Y}_1(\cdot)\bar{(\cdot)}} & R_{Y_{p-1}\bar{Y}_2(\cdot)\bar{(\cdot)}} & \cdots & R_{Y_{p-1}\bar{Y}_{p-1}(\cdot)\bar{(\cdot)}} & E^{p-1\bar{p}}_{(\cdot)\bar{(\cdot)}} \\ \overline{E^{1\bar{p}}}^{\mathsf{T}}_{(\cdot)\bar{(\cdot)}} & \overline{E^{2\bar{p}}}^{\mathsf{T}}_{(\cdot)\bar{(\cdot)}} & \cdots & \overline{E^{(p-1)\bar{p}}}^{\mathsf{T}}_{(\cdot)\bar{(\cdot)}} & Z_{(\cdot)\bar{(\cdot)}} \end{pmatrix},$$

and

$$A_3 = \begin{pmatrix} R_{X_1(\cdot)Y_1(\cdot)} & R_{X_1(\cdot)Y_2(\cdot)} & \cdots & R_{X_1(\cdot)Y_{p-1}(\cdot)} & R_{X_1(\cdot)Y_p(\cdot)} + P_{X_1(\cdot)(\cdot)} \\ R_{X_2(\cdot)Y_1(\cdot)} & R_{X_2(\cdot)Y_2(\cdot)} & \cdots & R_{X_2(\cdot)Y_{p-1}(\cdot)} & R_{X_2(\cdot)Y_p(\cdot)} + P_{X_2(\cdot)(\cdot)} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ R_{X_p(\cdot)Y_1(\cdot)} & R_{X_p(\cdot)Y_2(\cdot)} & \cdots & R_{X_p(\cdot)Y_{p-1}(\cdot)} & R_{X_p(\cdot)Y_p(\cdot)} + P_{X_p(\cdot)(\cdot)} \end{pmatrix},$$

where the tensor $Z_{\gamma\bar{\delta}}$ is defined as in (7.3):

$$\begin{aligned} E^{\mu\bar{p}}_{\gamma\bar{\delta}} &= R_{Y_\mu\bar{Y}_p\gamma\bar{\delta}} + P_{\gamma\bar{\delta}\bar{Y}_\mu}, \\ \overline{E^{\mu\bar{p}}}^{\mathsf{T}}_{\gamma\bar{\delta}} &= R_{Y_p\bar{Y}_\mu\gamma\bar{\delta}} + P_{\gamma\bar{\delta}\bar{Y}_\mu}, \quad 1 \le \mu \le p-1. \end{aligned}$$

Note $A_1 \ge 0$ since $\mathrm{Rm} \in \mathcal{C}_p$, and $A_2 \ge 0$ since $\mathcal{Q} \ge 0$. Now $\mathcal{J}$ can be written as

$$\mathcal{J} = \mathrm{trace}(A_1 \cdot A_2) + \left| P_{X_p\bar{\beta}\alpha} + \sum_{\mu=1}^{p} R_{X_\mu\bar{Y}_\mu\alpha\bar{\beta}} \right|^2 - \mathrm{trace}(A_3 \cdot \bar{A}_3).$$

Since $\mathcal{Q}(U \oplus W)$ achieves the minimum at (U, W) at time (x_0, t_0), then the second variation

$$\frac{\partial^2}{\partial s^2}\bigg|_{s=0} \mathcal{Q}(U(s) \oplus W(s)) \ge 0,$$

where $W(s) = W + sW_p$ and $U(s) = \sum_{\mu=1}^{p}(X_\mu + sW_\mu) \wedge \overline{(Y_\mu + sV_\mu)}$ for any $(1,0)$-type vectors $W_\mu, V_\mu \in T^{1,0}_{x_0}M$.

Through calculation, $\frac{\partial^2}{\partial s^2}|_{s=0}\mathcal{Q}(U(s)\oplus W(s)) \geq 0$ implies that

$$\begin{aligned}
&\sum_{\mu,\nu=1}^{p} R_{Y_\mu \bar{Y}_\nu \alpha\bar{\beta}} W_\nu^\alpha W_\mu^{\bar{\beta}} + \sum_{\mu=1}^{p}\big(P_{\alpha\bar{\beta}Y_\mu} W_p^\alpha W_\mu^{\bar{\beta}} + P_{\alpha\bar{\beta}\bar{Y}_\mu} W_\mu^\alpha W_p^{\bar{\beta}}\big)\\
&\quad + M_{\alpha\bar{\beta}} W_p^\alpha W_p^{\bar{\beta}} + \sum_{\mu=1}^{p}\big(P_{X_p\bar{\beta}\alpha} V_\mu^\alpha W_\mu^{\bar{\beta}} + P_{\alpha\bar{X}_p\bar{\beta}} W_\mu^\alpha V_\mu^{\bar{\beta}}\big)\\
&\quad + \sum_{\mu,\nu=1}^{p}\big(R_{Y_\mu\bar{X}_\mu\alpha\bar{\beta}} W_\nu^\alpha V_\nu^{\bar{\beta}} + R_{X_\mu\bar{Y}_\mu\alpha\bar{\beta}} V_\nu^\alpha W_\nu^{\bar{\beta}}\big)\\
&\quad + \sum_{\mu,\nu=1}^{p}\big(R_{X_\mu\bar{\alpha}Y_\nu\bar{\beta}} W_\nu^{\bar{\alpha}} V_\mu^{\bar{\beta}} + R_{\alpha\bar{X}_\mu\beta\bar{Y}_\nu} W_\nu^\alpha V_\mu^\beta\big)\\
&\quad + \sum_{\mu=1}^{p}\big(P_{X_\mu\bar{\alpha}\bar{\beta}} W_p^{\bar{\alpha}} V_\mu^{\bar{\beta}} + P_{\alpha\bar{X}_\mu\beta} W_p^\alpha V_\mu^\beta\big)\\
&\quad + \sum_{\mu,\nu=1}^{p} R_{X_\mu\bar{X}_\nu\alpha\bar{\beta}} V_\nu^\alpha V_\mu^{\bar{\beta}} \geq 0.
\end{aligned} \tag{7.8}$$

By letting $\mathcal{X} = \begin{pmatrix} W_1 \\ \vdots \\ W_p \end{pmatrix}$, $\mathcal{Y} = \begin{pmatrix} V_1 \\ \vdots \\ V_p \end{pmatrix}$, one can deduce from (7.8) that

$$\overline{\mathcal{X}}^{\mathsf{T}} A_2 \mathcal{X} + \overline{\mathcal{Y}}^{\mathsf{T}} A_1 \mathcal{Y} + 2\,\mathrm{Re}(\mathcal{Y}^{\mathsf{T}} \bar{A}_3 \mathcal{X} + \overline{\mathcal{Y}}^{\mathsf{T}} A_4 \mathcal{X}) \geq 0,$$

where

$$A_4 = \begin{pmatrix} G & 0 & \cdots & 0 \\ 0 & G & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & G \end{pmatrix},$$

where $G_{\alpha\bar{\beta}} = P_{\alpha\bar{X}_p\bar{\beta}} + \sum_{\mu=1}^{p} R_{Y_\mu\bar{X}_\mu\alpha\bar{\beta}}$.

If we regard $T^{1,0}_{x_0}M$ as $\mathbb{C}^m$, then $\mathcal{X}, \mathcal{Y} \in \mathbb{C}^{pm}$. By Lemma 7.4 below, which is due to Mok according to [8] (see also [13, lemma 2.86]), we have

$$\mathrm{trace}(A_2 \cdot A_1) \geq \mathrm{trace}(A_3 \cdot \bar{A}_3). \tag{7.9}$$

Inequality (7.9) implies that $\mathcal{J} \geq 0$. We then complete the proof of Theorem 7.1 for the case where M is compact. The case where M is noncompact will be treated in Section 10.

LEMMA 7.4 *Let $S(\mathcal{X}, \mathcal{Y})$ be a Hermitian symmetric quadratic form defined by*

$$S(\mathcal{X}, \mathcal{Y}) = A_{i\bar{j}} \mathcal{X}^i \overline{\mathcal{X}^j} + 2\,\mathrm{Re}(B_{ij} \mathcal{X}^i \mathcal{Y}^j + D_{i\bar{j}} \mathcal{X}^i \overline{\mathcal{Y}^j}) + C_{i\bar{j}} \mathcal{Y}^i \overline{\mathcal{Y}^j}.$$

If S is positive semidefinite, then

$$\sum_{i,j=1}^{N} A_{i\bar{j}}C_{j\bar{i}} \geq \max\left\{\sum_{i,j=1}^{N}|B_{ij}|^2, \sum_{i,j=1}^{N}|D_{i\bar{j}}|^2\right\}.$$

If one prefers notation without indices, the first three terms of (7.8) can be written as $\langle \mathcal{Q}(\sum_{\mu=1}^{p} Y_\mu \wedge \overline{W}_\mu), \overline{\sum_{\nu=1}^{p} Y_\nu \wedge \overline{W}_\nu}\rangle$. The last term can be written as $\langle \mathrm{Rm}(\sum_{\mu=1}^{p} X_\mu \wedge \overline{V}_\mu), \overline{\sum_{\nu=1}^{p} X_\nu \wedge \overline{V}_\nu}\rangle$.

8 Proof of Theorem 1.2

Before we start the proof, we remark that for the cases $p = 1$ and $p = m$ the result has been proved in [30, 32]. Hence we assume that $1 < p < m$. As before we deal with the compact case first and leave the noncompact case to a later section. By a perturbation argument it suffices to consider the case where ϕ is strictly positive. Then $\widetilde{Q} > 0$ for small t. Assume that at some point (x_0, t_0), $\widetilde{Q} = 0$ for the first time for some linearly independent vectors $v_1, \ldots, v_{p-1}$. As in Section 5, let $v_i(z)$ and $V(z)$ be variational vectors such that they depend on z holomorphically. Now consider the following function in z:

$$\begin{aligned}\mathcal{I}(z) \doteqdot{}& \frac{1}{2}[\mathrm{div}''(\mathrm{div}'(\phi)) + \mathrm{div}'(\mathrm{div}''(\phi))]_{v_1(z)\cdots v_{p-1}(z),\overline{v}_1(z)\cdots\overline{v}_{p-1}(z)}\\ &+ R_{j\bar{i}}\phi_{iv_1(z)\cdots v_{p-1}(z),\bar{j}\overline{v}_1(z)\cdots\overline{v}_{p-1}(z)}\\ &+ [\mathrm{div}'_{\overline{V}(z)}(\phi) + \mathrm{div}''_{V(z)}(\phi)]_{v_1(z)\cdots v_{p-1}(z),\overline{v}_1(z)\cdots\overline{v}_{p-1}(z)}\\ &+ \phi_{V(z)v_1(z)\cdots v_{p-1}(z),\overline{V}(z)\overline{v}_1(z)\cdots\overline{v}_{p-1}(z)} + \frac{\Lambda\phi_{v_1(z)\cdots v_{p-1}(z),\overline{v}_1(z)\cdots\overline{v}_{p-1}(z)}}{t},\end{aligned}$$

which satisfies $\mathcal{I}(0) = 0$ and $\mathcal{I}(z) \geq 0$ for any variational vectors $v_\mu(z)$ and $V(z)$ with $v_\mu(0) = v_\mu$ and $V(0) = V$. As before, without loss of generality we may assume that $\{v_1, \ldots, v_{p-1}\} = \{\frac{\partial}{\partial z^1}, \ldots, \frac{\partial}{\partial z^{p-1}}\}$. By the first variation consideration as in Section 5, we have that

$$\mathrm{div}'_{\overline{X}}(\phi) + \phi_{V,\overline{X}} = 0 = \mathrm{div}''_{X}(\phi) + \phi_{X,\overline{V}}, \tag{8.1}$$

$$\widetilde{Q}_{v_1\cdots(X)_\mu\cdots v_{p-1},\overline{v}_1\cdots\overline{v}_{p-1}} = 0 = \widetilde{Q}_{v_1\cdots v_{p-1},\overline{v}_1\cdots(\overline{X})_\nu\cdots v_{p-1}}, \tag{8.2}$$

for any $(1,0)$-type vectors X. By the second variation consideration we have that for any $(1,0)$-type vectors X and X_i,

$$\begin{aligned}
&\sum_{\mu,\nu=1}^{p-1} \widetilde{Q}_{v_1\cdots X_\mu\cdots v_{p-1},\overline{v}_1\cdots\overline{X}_\nu\cdots\overline{v}_{p-1}} + \phi_{Xv_1\cdots v_{p-1},\overline{X}\overline{v}_1\cdots\overline{v}_{p-1}} \\
&\quad + \sum_{\mu=1}^{p-1} \operatorname{div}'_{\overline{X}}(\phi)_{v_1\cdots X_\mu\cdots v_{p-1},\overline{v}_1\cdots\overline{v}_{p-1}} + \phi_{Vv_1\cdots X_\mu\cdots v_{p-1},\overline{X}\overline{v}_1\cdots\overline{v}_{p-1}} \\
&\quad + \sum_{\nu=1}^{p-1} \operatorname{div}''_X(\phi)_{v_1\cdots v_{p-1},\overline{v}_1\cdots\overline{X}_\nu\cdots\overline{v}_{p-1}} + \phi_{Xv_1\cdots v_{p-1},\overline{V}\overline{v}_1\cdots\overline{X}_\nu\cdots\overline{v}_{p-1}} \\
&\quad \geq 0.
\end{aligned}$$

This implies that the block matrix (as defined in Section 5) $\mathcal{M}_1 \geq 0$. However here Q is replaced by $\widetilde{Q}$.

To check that $(\frac{\partial}{\partial t} - \Delta)\widetilde{Q}_{1\cdots p-1,\overline{1}\cdots\overline{p-1}} \geq 0$ we may extend V such that the following equations are satisfied:

$$\left(\frac{\partial}{\partial t} - \Delta\right)V^i = -\frac{1}{t}V^i, \quad \nabla_i V = R_i^j \frac{\partial}{\partial z^j} + \frac{1}{t}\frac{\partial}{\partial z^i}, \quad \nabla_{\overline{i}} V^j = 0.$$

Using this set of equations, (6.7), (6.8), and (4.20) can be simplified to

$$\begin{aligned}
&\left(\frac{\partial}{\partial t} - \Delta\right)(\operatorname{div}''_V(\phi))_{I_{p-1},\overline{J}_{p-1}} \\
&\quad = -\frac{1}{t}\operatorname{div}''_V(\phi) + \mathcal{KB}(\operatorname{div}''_V(\phi)) \\
&\qquad + \sum_{\nu=1}^{p-1} R^{k\overline{l}}_{V\overline{j}_\nu}(\operatorname{div}''(\phi))_{ki_1\cdots i_{p-1},\overline{j}_1\cdots(\overline{l})_\nu\cdots\overline{j}_{p-1}} \\
&\qquad - \frac{1}{2}\operatorname{div}''_{\operatorname{Ric}(V)}(\phi) - \frac{1}{t}\operatorname{div}'(\operatorname{div}''(\phi)) \\
&\qquad + R_{j\overline{k}}\nabla_k\phi_{VI_{p-1},\overline{jJ}_{p-1}} + \nabla_V R_{j\overline{i}}\phi_{iI_{p-1},\overline{jJ}_{p-1}} \\
&\qquad + \sum_{\mu=1}^{p-1} \nabla_{i_\mu} R_{j\overline{k}}\phi_{Vi_1\cdots(k)_\mu\cdots i_{p-1},\overline{jJ}_{p-1}} - R_{l\overline{k}}\nabla_{\overline{l}}(\operatorname{div}''(\phi))_{kI_{p-1},\overline{J}_{p-1}},
\end{aligned} \tag{8.3}$$

$$
\begin{aligned}
&\left(\frac{\partial}{\partial t}-\Delta\right)(\mathrm{div}'_{\overline{V}}(\phi))_{I_{p-1},\overline{J}_{p-1}}\\
&\quad=-\frac{1}{t}\,\mathrm{div}'_{\overline{V}}(\phi)+\mathcal{KB}(\mathrm{div}'_{\overline{V}}(\phi))\\
&\qquad+\sum_{\mu=1}^{p-1}R^{k\overline{l}}_{i_\mu\overline{V}}(\mathrm{div}'(\phi))_{i_1\cdots(k)_\mu\cdots i_{p-1},\overline{l}\,\overline{j}_1\cdots\overline{j}_{p-1}}\\
&\qquad-\frac{1}{2}\,\mathrm{div}'_{\overline{\mathrm{Ric}(V)}}(\phi)-\frac{1}{t}\,\mathrm{div}''(\mathrm{div}'(\phi))+R_{k\overline{i}}\nabla_{\overline{k}}\phi_{iI_{p-1},\overline{VJ}_{p-1}}\\
&\qquad+\nabla_{\overline{V}}R_{j\overline{i}}\phi_{iI_{p-1},\overline{jJ}_{p-1}}+\sum_{\nu=1}^{p-1}\nabla_{\overline{j}_\nu}R_{l\overline{i}}\phi_{iI_{p-1},\overline{V}\,\overline{j}_1\cdots(\overline{l})_\nu\cdots\overline{j}_{p-1}}\\
&\qquad-R_{l\overline{k}}\nabla_k(\mathrm{div}'(\phi))_{I_{p-1},\overline{lJ}_{p-1}},
\end{aligned}
\tag{8.4}
$$

$$
\begin{aligned}
&\left(\frac{\partial}{\partial t}-\Delta\right)\phi_{V,\overline{V}}\\
&\quad=\mathcal{KB}(\phi_{V,\overline{V}})+R^{k\overline{l}}_{V\,\overline{V}}\phi_{k,\overline{l}}+\sum_{\nu=1}^{p-1}R^{k\overline{l}}_{V\,\overline{j}_\nu}\phi_{kI_{p-1},\overline{V}\,\overline{j}_1\cdots(\overline{l})_\nu\cdots\overline{j}_{p-1}}\\
&\qquad+\sum_{\mu=1}^{p-1}R^{k\overline{l}}_{i_\mu\,\overline{V}}\phi_{Vi_1\cdots(k)_\mu\cdots i_{p-1},\overline{l}\,\overline{J}_{p-1}}-\frac{1}{2}(\phi_{V,\overline{\mathrm{Ric}(V)}}+\phi_{\mathrm{Ric}(V),\overline{V}})\\
&\qquad-\frac{2}{t}\phi_{V,\overline{V}}-\frac{\Lambda\phi}{t^2}-\frac{2}{t}R_{j\overline{i}}\phi_{iI_{p-1},\overline{jJ}_{p-1}}-\frac{1}{t}\,\mathrm{div}'_{\overline{V}}(\phi)-\frac{1}{t}\,\mathrm{div}''_{V}(\phi)\\
&\qquad-(R_{j\overline{k}}\nabla_k\phi_{VI_{p-1},\overline{jJ}_{p-1}}+R_{k\overline{i}}\nabla_{\overline{k}}\phi_{iI_{p-1},\overline{V}\,\overline{J}_{p-1}})\\
&\qquad-R_{j\overline{k}}R_{k\overline{i}}\phi_{iI_{p-1},\overline{jJ}_{p-1}}.
\end{aligned}
\tag{8.5}
$$

Adding (8.3), (8.4), and (8.5) with the two evolution equations in Lemma 6.4 and (6.6) and (6.9), and using (8.1), we have that

$$
\begin{aligned}
&\left(\frac{\partial}{\partial t}-\Delta\right)\widetilde{Q}_{I_{p-1},\overline{J}_{p-1}}\\
&\quad=\mathcal{KB}(\widetilde{Q})_{I_{p-1},\overline{J}_{p-1}}+\sum_{i\,j=1}^{m}Z_{j\overline{i}}\phi_{iI_{p-1},\overline{jJ}_{p-1}}\\
&\qquad+\sum_{\mu=1}^{p-1}(R_{i_\mu\overline{V}l\overline{k}}+P_{l\overline{k}i_\mu})\big(\mathrm{div}'(\phi)_{i_1\cdots(k)_\mu\cdots i_{p-1},\overline{lJ}_{p-1}}+\phi_{Vi_1\cdots(k)_\mu\cdots i_{p-1},\overline{lJ}_{p-1}}\big)+
\end{aligned}
$$

$$+\sum_{\nu=1}^{p-1}(R_{V\bar{j}_\nu l\bar{k}}+P_{l\bar{k}\bar{j}_\nu})\big(\mathrm{div}''(\phi)_{kI_{p-1},\bar{j}_1\cdots(\bar{l})_\nu\cdots\bar{j}_{p-1}}+\phi_{kI_{p-1},\bar{V}\bar{j}_1\cdots(\bar{l})_\nu\cdots\bar{j}_{p-1}}\big)$$
$$-\frac{2}{t}\widetilde{Q}_{I_{p-1}\overline{J}_{p-1}}.$$

Now the nonnegativity of $(\frac{\partial}{\partial t}-\Delta)\widetilde{Q}_{1\cdots(p-1),\overline{1}\cdots(\overline{p-1})}$ at (x_0,t_0) can be proved in a similar way as in the argument in Section 2. Observe that the part involving only Ric in $\mathcal{KB}(\tilde{Q})_{1\cdots(p-1),\overline{1}\cdots(\overline{p-1})}$ is

$$-\frac{1}{2}\sum_{i=1}^{p-1}\big(\widetilde{Q}_{1\cdots\mathrm{Ric}(i)\cdots(p-1),\overline{1}\cdots(\overline{p-1})}+\widetilde{Q}_{1\cdots(p-1),\overline{1}\cdots\overline{\mathrm{Ric}(i)}\cdots(\overline{p-1})}\big),$$

which vanishes due to (8.2). Hence we only need to establish the nonnegativity of

$$\begin{aligned}\tilde{J}\doteq&\sum_{\mu=1}^{p-1}\sum_{\nu=1}^{p-1}R_{\mu\bar{\nu}l\bar{k}}\widetilde{Q}_{1\cdots(k)_\mu\cdots(p-1),\overline{1}\cdots(\bar{l})_\nu\cdots(\overline{p-1})}\\&+\sum_{\mu=1}^{p-1}(R_{\mu\bar{V}l\bar{k}}+P_{l\bar{k}\mu})\big(\mathrm{div}'(\phi)_{1\cdots(k)_\mu\cdots(p-1),\overline{l}\overline{J}_{p-1}}+\phi_{V1\cdots(k)_\mu\cdots(p-1),\overline{l}\overline{J}_{p-1}}\big)\\&+\sum_{\nu=1}^{p-1}(R_{V\bar{\nu}l\bar{k}}+P_{l\bar{k}\bar{\nu}})(\mathrm{div}''(\phi)_{kI_{p-1},\overline{1}\cdots(\bar{l})_\nu\cdots\overline{p-1}}+\phi_{kI_{p-1},\bar{V}\overline{1}\cdots(\bar{l})_\nu\cdots\overline{p-1}})\\&+\sum_{i,j=1}^{m}Z_{j\bar{i}}\phi_{i1\cdots(p-1),\overline{j1}\cdots(\overline{p-1})}.\end{aligned}$$

By Theorem 7.1, the assumption that the curvature operator Rm is in $\mathcal{C}_p$ implies that the matrix

$$\mathcal{M}_3=\begin{pmatrix}R_{1\overline{1}(\cdot)\overline{(\cdot)}}&R_{1\overline{2}(\cdot)\overline{(\cdot)}}&\cdots&R_{1\overline{p-1}(\cdot)\overline{(\cdot)}}&D^1_{(\cdot)\overline{(\cdot)}}\\R_{2\overline{1}(\cdot)\overline{(\cdot)}}&R_{2\overline{2}(\cdot)\overline{(\cdot)}}&\cdots&R_{2\overline{p-1}(\cdot)\overline{(\cdot)}}&D^2_{(\cdot)\overline{(\cdot)}}\\\vdots&\vdots&\ddots&\vdots&\vdots\\R_{p-1\overline{1}(\cdot)\overline{(\cdot)}}&R_{p-1\overline{2}(\cdot)\overline{(\cdot)}}&\cdots&R_{p-1\overline{p-1}(\cdot)\overline{(\cdot)}}&D^{p-1}_{(\cdot)\overline{(\cdot)}}\\\overline{D^1}^{\mathsf{T}}_{(\cdot)\overline{(\cdot)}}&\overline{D^2}^{\mathsf{T}}_{(\cdot)\overline{(\cdot)}}&\cdots&\overline{D^{p-1}}^{\mathsf{T}}_{(\cdot)\overline{(\cdot)}}&Z_{(\cdot)\overline{(\cdot)}}\end{pmatrix}\geq 0,$$

where $D^i_{\mu\bar{\nu}}=R_{i\bar{V}\mu\bar{\nu}}+P_{\mu\bar{\nu}i}$. The nonnegativity of $\tilde{J}$ follows from $\mathrm{trace}(\mathcal{M}_1\cdot\mathcal{M}_3)\geq 0$. Here $\mathcal{M}_1$ is the block matrix in Section 5. This proves Theorem 6.1 for the case where M is compact. We postpone the proof of the noncompact case to a later section.

9 LYH Type Estimates for Ricci Flow under Condition $\widetilde{\mathcal{C}}_p$

In this section we prove another set of LYH type estimates for the Ricci flow. Let $(M, g(t))$ be a complete solution to

$$\frac{\partial}{\partial t} g_{ij} = -2R_{ij}. \tag{9.1}$$

Recall that Hamilton proved that if $\mathrm{Rm} \geq 0$ and bounded, then the quadratic form

$$\widetilde{\mathcal{Q}}(W \oplus U) \doteq \langle \mathcal{M}(W), W\rangle + 2\langle P(W), U\rangle + \langle \mathrm{Rm}(U), U\rangle \geq 0$$

where the $\mathcal{M}$ and P are defined in a normal frame by

$$\begin{aligned} \mathcal{M}_{ij} &\doteq \Delta R_{ij} - \frac{1}{2}\nabla_i\nabla_j R + 2R_{ikjl}R_{kl} - R_{ik}R_{jk} + \frac{1}{2t}R_{ij}, \\ P_{ijk} &\doteq \nabla_i R_{jk} - \nabla_j R_{ik}, \end{aligned}$$

with $\langle P(W), U\rangle = P_{ijk}W^k U^{ij}$. One can view $\widetilde{\mathcal{Q}}$ as the restriction of a Hermitian quadratic form

$$\langle \widetilde{\mathcal{Q}}(W \oplus U), \overline{W \oplus U}\rangle \doteq \langle \mathcal{M}(W), \overline{W}\rangle + 2\,\mathrm{Re}\langle P(W), \overline{U}\rangle + \langle \mathrm{Rm}(U), \overline{U}\rangle,$$

which is defined on $\bigwedge^2(\mathbb{C}^n) \oplus \mathbb{C}^n$. We also denote

$$\langle \mathcal{Z}(W \wedge Z), \overline{W \wedge Z}\rangle \doteq \widetilde{\mathcal{Q}}(W \oplus (W \wedge Z)).$$

If we fix a Z, $\mathcal{Z}$ can be viewed as a Hermitian bilinear form of W, which we denote by $\mathcal{Z}_Z$, or still by $\mathcal{Z}$ when the meaning is clear. In terms of a local frame, it can be written as

$$(\mathcal{Z}_Z)_{cd} = \mathcal{M}_{cd} + P_{dac}\overline{Z}^a + P_{cad}Z^a + R_{Zc\overline{Z}d}.$$

THEOREM 9.1 *Assume that $(M, g(t))$ on $M \times [0, T]$ satisfies $\widetilde{\mathcal{C}}_p$. When M is noncompact we also assume that the curvature of $(M, g(t))$ is uniformly bounded on $M \times [0, T]$. Then for any $t > 0$, $\widetilde{\mathcal{Q}} \geq 0$ for any $(x, t) \in M \times [0, T]$, $W \in T_xM \otimes \mathbb{C}$, and $U \in \bigwedge^2(T_xM \otimes \mathbb{C})$ such that $U = \sum_{\mu=1}^p W_\mu \wedge Z_\mu$ with $W_p = W$. Furthermore, the equality holds for some $t > 0$; the universal cover of $(M, g(t))$ is a gradient-expanding Ricci soliton.*

Remark 9.2. In [5], a slightly weaker result was proved for the $p = 1$ case. As before, for p large enough the condition $\widetilde{\mathcal{C}}_p$ is equivalent to $\mathrm{Rm} \geq 0$ and the above result is equivalent to Hamilton's theorem. Hence our result gives a family of estimates interpolating between those of [5] and [17].

In theorem 4.1 of [17], the following result was proved by brutal force computations:

LEMMA 9.3 *At (x_0, t_0), if W and U are extended by the equations,*

$$\left(\frac{\partial}{\partial t} - \Delta\right)W = \frac{W}{t} + \mathrm{Ric}(W), \quad \nabla W = 0,$$

$$\left(\frac{\partial}{\partial t}-\Delta\right)U^{ab}=R_c^aU^{cb}+R_c^bU^{ac},$$

$$\nabla_aU^{bc}=\frac{1}{2}\big(R_a^bW^c-R_a^cW^b\big)+\frac{1}{4t}\big(g_a^bW^c-g_a^cW^b\big),$$

then under an orthonormal frame

$$\begin{aligned}\left(\frac{\partial}{\partial t}-\Delta\right)\widetilde{\mathcal{Q}}&=2R_{acbd}\mathcal{M}_{cd}W^a\overline{W}^b-2P_{acd}P_{bdc}W^a\overline{W}^b\\&\quad+8\,\mathrm{Re}\big(R_{adce}P_{dbe}W^c\overline{U}^{ab}\big)+4R_{aecf}R_{bedf}U^{ab}\overline{U}^{cd}\\&\quad+|P(W)+\mathrm{Rm}(U)|^2.\end{aligned}\tag{9.2}$$

Here R_b^a denotes the Ric *transformation in terms of the local frame.*

Using the notation of [16], the term $4R_{aecf}R_{bedf}U^{ab}\overline{U}^{cd}$ can be expressed as $8\langle\mathrm{Rm}^{\#}(U),\overline{U}\rangle$.

Assume that $\widetilde{\mathcal{Q}}\geq 0$ for $M\times[0,t_0]$ and at (x_0,t_0) it vanishes for $W\oplus U$, with $U=\sum_{\mu=1}^p W_\mu\wedge Z_\mu$, and $W_p=W$. Now let $W_\mu(z)$ and $Z_\mu(z)$ be a variation of W_μ and Z_μ with $W_\mu(z)=W_\mu+zX_\mu$ and $Z_\mu(z)=Z_\mu+zY_\mu$. Let $\widetilde{\mathcal{I}}(z)\doteqdot\widetilde{\mathcal{Q}}(W(z)\oplus U(z))$ with $U(z)=\sum_{\mu=1}^p W_\mu(z)\wedge Z_\mu(z)$. Using $\Delta\widetilde{\mathcal{I}}(0)\geq 0$, we deduce the following estimate:

$$\begin{aligned}&\sum_{\mu,\nu=1}^p R_{X_\mu Z_\mu\overline{X}_\nu\overline{Z}_\nu}+2\,\mathrm{Re}\left(\left\langle P(X_p),\overline{\sum_{\mu=1}^p X_\mu\wedge Z_\mu}\right\rangle\right)+\langle\mathcal{M}(X_p),\overline{X}_p\rangle\\&\quad+2\,\mathrm{Re}\left(\sum_{\mu,\nu=1}^p R_{X_\mu Z_\mu\overline{W}_\nu\overline{Y}_\nu}\right)+2\,\mathrm{Re}\left(\left\langle P(X_p),\overline{\sum_{\mu=1}^p W_\mu\wedge Y_\mu}\right\rangle\right)\\&\quad+\sum_{\mu,\nu=1}^p R_{W_\mu Y_\mu\overline{W}_\nu\overline{Y}_\nu}\geq 0.\end{aligned}\tag{9.3}$$

To prove Theorem 9.1 for the compact case, it suffices to show that the right-hand side of (9.2) is nonnegative for a null vector $W\oplus U$ with $U=\sum_{\mu=1}^p W_\mu\wedge Z_\mu$ and $W_p=W$. Denote the first four terms in the right-hand side of (9.2) by $\widetilde{\mathcal{J}}$. Expand it and let $\widehat{P}_{dc}(Z_p)=P_{dac}Z_p^a$. We then obtain that

$$\begin{aligned}\widetilde{\mathcal{J}}&=2R_{W_pc\overline{W}_pd}\mathcal{Z}_{cd}-2R_{W_pc\overline{W}_pd}(\widehat{P}_{dc}(\overline{Z}_p)+\widehat{P}_{cd}(Z_p))\\&\quad-2R_{W_pc\overline{W}_pd}R_{Z_pc\overline{Z}_pd}\\&\quad+2\sum_{\mu,\nu=1}^p(R_{W_\mu e\overline{W}_\nu f}R_{Z_\mu e\overline{Z}_\nu f}-R_{W_\mu e\overline{Z}_\nu f}R_{Z_\mu e\overline{W}_\nu f})+\end{aligned}$$

$$+ 4\,\mathrm{Re}\Bigg(\sum_{\mu=1}^{p-1} R_{\overline{W}_\mu d W_p e}\,\widehat{P}_{de}(\overline{Z}_\mu)\Bigg) + 4\,\mathrm{Re}(R_{\overline{W}_p d W_p e}\,\widehat{P}_{de}(\overline{Z}_p))$$
$$- 4\,\mathrm{Re}\Bigg(\sum_{\mu=1}^{p-1} R_{\overline{Z}_\mu d W_p e}\,\widehat{P}_{de}(\overline{W}_\mu)\Bigg) - 4\,\mathrm{Re}(R_{\overline{Z}_p d W_p e}\,\widehat{P}_{de}(\overline{W}_p))$$
$$- 2\widehat{P}_{de}(W_p)\widehat{P}_{ed}(\overline{W}_p).$$

After some cancellations (the second term and the seventh term on the right-hand side above cancel each other) we can rewrite the above as $\widetilde{\mathcal{J}} = 2(\mathrm{trace}(B_1 B_2) - \mathrm{trace}(B_3 \cdot \overline{B}_3))$ where

$$B_1 = \begin{pmatrix} R_{W_1(\cdot)\overline{W}_1(\cdot)} & R_{W_1(\cdot)\overline{W}_2(\cdot)} & \cdots & R_{W_1(\cdot)\overline{W}_p(\cdot)} \\ R_{W_2(\cdot)\overline{W}_1(\cdot)} & R_{W_2(\cdot)\overline{W}_2(\cdot)} & \cdots & R_{W_2(\cdot)\overline{W}_p(\cdot)} \\ \vdots & \vdots & \ddots & \vdots \\ R_{W_p(\cdot)\overline{W}_1(\cdot)} & R_{W_p(\cdot)\overline{W}_2(\cdot)} & \cdots & R_{W_p(\cdot)\overline{W}_p(\cdot)} \end{pmatrix},$$

$$B_2 = \begin{pmatrix} R_{\overline{Z}_1(\cdot)Z_1(\cdot)} & R_{\overline{Z}_1(\cdot)Z_2(\cdot)} & \cdots & R_{\overline{Z}_1(\cdot)Z_{p-1}(\cdot)} & F^{1p} \\ R_{\overline{Z}_2(\cdot)Z_1(\cdot)} & R_{\overline{Z}_2(\cdot)Z_2(\cdot)} & \cdots & R_{\overline{Z}_2(\cdot)Z_{p-1}(\cdot)} & F^{2p} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ R_{\overline{Z}_{p-1}(\cdot)Z_1(\cdot)} & R_{\overline{Z}_{p-1}(\cdot)Z_2(\cdot)} & \cdots & R_{\overline{Z}_{p-1}(\cdot)Z_{p-1}(\cdot)} & F^{p-1p} \\ \overline{F^{1p}}^{\mathsf{T}} & \overline{F^{2p}}^{\mathsf{T}} & \cdots & \overline{F^{(p-1)p}}^{\mathsf{T}} & \mathcal{Z}_{(\cdot)(\cdot)} \end{pmatrix},$$

and

$$B_3 = \begin{pmatrix} R_{W_1(\cdot)\overline{Z}_1(\cdot)} & R_{W_1(\cdot)\overline{Z}_2(\cdot)} & \cdots & R_{W_1(\cdot)\overline{Z}_{p-1}(\cdot)} & R_{W_1(\cdot)\overline{Z}_p(\cdot)} + \widehat{P}(W_1) \\ R_{W_2(\cdot)\overline{Z}_1(\cdot)} & R_{W_2(\cdot)\overline{Z}_2(\cdot)} & \cdots & R_{W_2(\cdot)\overline{Z}_{p-1}(\cdot)} & R_{W_2(\cdot)\overline{Z}_p(\cdot)} + \widehat{P}(W_2) \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ R_{W_p(\cdot)\overline{Z}_1(\cdot)} & R_{W_p(\cdot)\overline{Z}_2(\cdot)} & \cdots & R_{W_p(\cdot)\overline{Z}_{p-1}(\cdot)} & R_{W_p(\cdot)\overline{Z}_p(\cdot)} + \widehat{P}(W_p) \end{pmatrix},$$

with $F^{\mu p} = R_{\overline{Z}_\mu(\cdot)Z_p(\cdot)} + \widehat{P}_{(\cdot)(\cdot)}(\overline{Z}_\mu)$.

On the other hand, using the above notation, we can rewrite (9.3) as

$$\mathcal{Z}_{X_p\overline{X}_p} + 2\sum_{\mu=1}^{p-1}\mathrm{Re}(R_{\overline{Z}_\mu\overline{X}_\mu Z_p X_p}) + 2\sum_{\mu=1}^{p-1}\mathrm{Re}(\widehat{P}_{\overline{X}_\mu X_p}(\overline{Z}_\mu))$$
$$+ \sum_{\mu,\nu=1}^{p-1} R_{\overline{Z}_\mu\overline{X}_\mu Z_\nu X_\nu} - 2\sum_{\mu=1}^{p}\mathrm{Re}\Bigg(\widehat{P}_{\overline{Y}_\mu X_p}(\overline{W}_\mu) + \sum_{\nu=1}^{p} R_{\overline{W}_\mu\overline{Y}_\mu Z_\nu X_\nu}\Bigg)$$
$$+ \sum_{\mu,\nu=1}^{p} R_{W_\mu Y_\mu\overline{W}_\nu\overline{Y}_\nu} \geq 0,$$

which implies that $S(\mathcal{X}, \mathcal{Y})$ is nonnegative, where

$$S(\mathcal{X}, \mathcal{Y}) = (B_2)_{ij}\mathcal{X}^i\overline{\mathcal{X}^j} - 2\,\mathrm{Re}((B_3)_{ij}\mathcal{Y}^i\overline{\mathcal{X}^j}) + (B_1)_{ij}\mathcal{Y}^i\overline{\mathcal{Y}^j}$$

with $\mathcal{X} = (X_1 \cdots X_p)^{\mathsf{T}}$, $\mathcal{Y} = (Y_1 \cdots Y_p)^{\mathsf{T}}$. Hence by Lemma 7.4 we can conclude that $\widetilde{\mathcal{J}} \geq 0$ and complete the proof of Theorem 9.1 for the compact case. The case that M is noncompact will be proved in the next section together with Theorem 7.1.

10 Complete Noncompact Manifolds with Bounded Curvature

In this section we first show that under the condition that the curvature tensor of $(M, g(t))$, a solution to the Ricci flow or Kähler-Ricci flow, is uniformly bounded on $M \times [0, T]$, the maximum principle can still apply, and we conclude the invariance of the cones $\mathcal{C}_p$, $\widetilde{\mathcal{C}}_p$ from Section 3. Moreover, the LYH type estimates for the Kähler-Ricci flow and the Ricci flow in Sections 7 and 9 remain valid.

First we show the invariance of $\mathcal{C}_p$ and $\widetilde{\mathcal{C}}_p$. In fact, the following maximum principle holds on noncompact manifolds. Consider V, a vector bundle over M, with a fixed metric $\widetilde{h}$, a time-dependent metric connection $D^{(t)}$. On M there are time-dependent metrics $g(t)$ and $\nabla^{(t)}$, the Levi-Civita connection of $g(t)$. When the meaning is clear we often omit the superscript (t). The main concern of this subsection is the diffusion-reaction equation

$$\begin{cases} \frac{\partial}{\partial t} f(x,t) - \Delta f(x,t) = \Phi(f)(x,t), \\ f(x,0) = f_0(x). \end{cases} \tag{10.1}$$

Here $\Delta = g^{ij}(x,t) D_i D_j$. We know that after applying Uhlenbeck's trick [16], the study of the curvature operator under the Ricci flow equation is a subcase of this general formulation. One can modify the proof of theorem 1.1 in [4] to obtain the following result:

THEOREM 10.1 *Assume that M is a complete noncompact manifold and Φ is locally Lipschitz. Let $(M, g(t))$ be a solution to Ricci flow such that $|\mathrm{Rm}|(x,t) \leq A$ for some $A > 0$ for any $(x,t) \in M \times [0,T]$. Let $C(t) \subset V$, $t \in [0,T]$, be a family of closed, full-dimensional cones, depending continuously on t. Suppose that each of the cones $C(t)$ is invariant under parallel transport and fiberwise convex and that the family $\{C(t)\}$ is preserved by the ODE $\frac{d}{dt} f(t) = \Phi(f)$. Moreover, assume that there exists a smooth section* I *that is invariant under the parallel transport and* $\mathrm{I} \in C(t)$ *for all t. If $f(x,t)$ satisfies* (10.1) *with $f(x,0) \in C(0)$, $|f|(x,t) \leq B$ on $M \times [0,T]$ for some $B > 0$, then $f(x,t) \in C(t)$ for $(x,t) \in M \times [0,T]$.*

PROOF. The key is Lemma 10.2 below, which ensures the existence of a smooth function φ such that $\varphi(x,t) \to +\infty$ uniformly on $[0,\eta]$ for some $\eta > 0$ and

$$\left(\frac{\partial}{\partial t} - \Delta\right)\varphi \geq C\varphi.$$

Clearly once we can prove the result for $[0,\eta]$, we can iterate the procedure and get the result on $[0,T]$.

For any $\epsilon > 0$, we can fix a compact region K such that $\tilde{f}(x,t) \doteqdot f(x,t) + \epsilon\varphi \mathrm{I} \in C(t)$ for all (x,t) with $x \in M \setminus K$. In fact, one can choose $K = \overline{B^0(p, R_0)}$, a closed ball of a certain radius R_0 with respect to the initial metric. Now for every t, $\rho(x,t) = \mathrm{dist}^2(\tilde{f}(x,t), C_x(t))$ with $C_x(t) = C(t) \cap V_x$ achieves a maximum somewhere. The argument of [4] can be applied, and we only need to restrict ourselves over $K \times [0, \eta]$. In particular, we let $\rho(t) = \rho(x_0, t) = \max \rho(\cdot, t)$.

Since Φ is locally Lipschitz, it is easy to infer that there exists A' such that $|\tilde{f}| + |\Phi(\tilde{f})| \leq A'$ for some constant A' on $K \times [0,T]$. Since $\varphi > 0$, we can choose C large enough such that $\epsilon C\varphi \mathrm{I} + \Phi(f) - \Phi(\tilde{f}) \in C(t)$ for all $(x,t) \in K \times [0,\eta]$. Now the rest of the argument in [4] can be invoked to conclude that $D_-\rho(t) \leq L\rho(t)$ with L depending on the local Lipschitz constant of Φ. Here D_- is the lower Dini's derivative from the left.

Specifically, we have

$$\begin{aligned} D_-\rho(t) &\leq 2\left\langle \frac{\partial}{\partial t}\tilde{f}, \tilde{f} - v_\infty \right\rangle\Big|_{(x_0,t)} - 2\langle \Phi(v_\infty), \tilde{f}(x_0,t) - v_\infty \rangle \\ &= 2\langle (\Delta \tilde{f})(x_0,t), \tilde{f}(x_0,t) - v_\infty \rangle \\ &\quad + 2\langle \Phi(f) + \epsilon\left(\frac{\partial}{\partial t} - \Delta\right)\varphi \mathrm{I}\Big|_{(x_0,t)} - \Phi(v_\infty), \tilde{f}(x_0,t) - v_\infty \rangle. \end{aligned}$$

Here v_∞ is a vector in V_{x_0} such that $\mathrm{dist}(\tilde{f}(x_0,t), v_\infty) = \mathrm{dist}(\tilde{f}(x_0,t), C_{x_0}(t))$. By lemma 1.2 of [4],

$$\langle (\Delta \tilde{f})(x_0,t), \tilde{f}(x_0,t) - v_\infty \rangle \leq 0.$$

For sufficiently large C, $\epsilon C\varphi \mathrm{I} + \Phi(f) - \Phi(\tilde{f}) \in C(t)$, which implies that

$$\langle \epsilon C\varphi \mathrm{I} + \Phi(f) - \Phi(\tilde{f}), \tilde{f}(x_0,t) - v_\infty \rangle \leq 0.$$

Hence by the convexity of $C(t)$,

$$\left\langle \Phi(f) + \epsilon\left(\frac{\partial}{\partial t} - \Delta\right)\varphi \mathrm{I}\Big|_{(x_0,t)} - \Phi(\tilde{f}), \tilde{f}(x_0,t) - v_\infty \right\rangle \leq 0.$$

Combining the above we conclude that

$$D_-\rho(t) \leq 2\langle \Phi(\tilde{f}(x_0,t)) - \Phi(v_\infty), \tilde{f}(x_0,t) - v_\infty \rangle \leq L\rho(t).$$

The rest of the proof follows from [4] verbatim. □

LEMMA 10.2 *Assume that M is a complete, noncompact manifold. Let $(M, g(t))$ be a solution to Ricci flow such that $|\mathrm{Rm}|(x,t) \leq A$ for some $A > 0$ for any $(x,t) \in M \times [0,T]$. Then there exist $C_1 > 0$ and a positive function $\varphi(x,t)$ such that for any given $C > 0$, there exists $\eta > 0$ such that on $M \times [0,\eta]$*

$$\exp(C_1^{-1}(r_0(x) + 1)) \leq \varphi(x,t) \leq \exp(C_1(r_0(x) + 1) + 1),$$

$$\left(\frac{\partial}{\partial t} - \Delta\right)\varphi \geq C\varphi.$$

Here $r_0(x)$ is the distance to a fixed point with respect to the initial metric.

PROOF. First by lemma 5.1 of [17], there exist $f(x)$ such that

$$C_1^{-1}(1 + r_0(x)) \le f(x) \le C_1(1 + r_0(x)),$$
$$|\nabla f|^2 + |\nabla^2 f| \le C_1.$$

Now we let $\varphi = \exp(\alpha t + f(x))$. By choosing a suitable $\alpha > 0$, the claimed result follows easily. □

COROLLARY 10.3 *Let $(M, g(t))$ be a solution to Ricci flow (or Kähler-Ricci flow) such that $|\mathrm{Rm}|(x,t) \le A$ for some $A > 0$ for any $(x,t) \in M \times [0,T]$. Then $\widetilde{\mathcal{C}}_p$ is invariant under the Ricci flow (respectively, $\mathcal{C}_p$ is invariant under the Kähler-Ricci flow).*

Concerning the LYH type estimates for the Ricci flow and Kähler-Ricci flow, we can evoke the perturbation argument of Hamilton [17]. Note that by passing to $[\epsilon, T - \epsilon]$, the curvature bound, due to Shi's derivative estimates, implies that all the derivatives of the curvature are uniformly bounded. Now consider the perturbed quantity

$$\widetilde{\mathcal{Q}}'(W \oplus U) = \langle \mathcal{M}(W), \overline{W} \rangle + \frac{\varphi}{t}\langle W, \overline{W} \rangle + 2\,\mathrm{Re}(P(W), \overline{U}\rangle)$$
$$+ \langle \mathrm{Rm}(U), \overline{U} \rangle + \psi |U|^2$$

where φ and ψ are the functions from lemma 5.2 of [17]. Following the argument of section 5 in [17] verbatim, we can show the following result.

COROLLARY 10.4 *Assume that $(M, g(t))$ is a solution to the Ricci flow on $M \times [0,T]$ such that $|\mathrm{Rm}|(x,t) \le A$. Assume that $\mathrm{Rm}(g(x,0)) \in \widetilde{\mathcal{C}}_p$. Then for any $t > 0$, $\widetilde{\mathcal{Q}} \ge 0$ for any $(x,t) \in M \times [0,T]$, $W \in T_x M \otimes \mathbb{C}$, and $U \in \bigwedge^2 (T_x M \otimes \mathbb{C})$ such that $U = \sum_{\mu=1}^{p} W_\mu \wedge Z_\mu$ with $W_p = W$ (respectively, if $(M, g(t))$ is a solution to the Kähler-Ricci flow with $\mathrm{Rm}(g(x,0)) \in \mathcal{C}_p$, then $\mathcal{Q} \ge 0$).*

11 Complete Noncompact Manifolds without Curvature Bound

We first discuss the existence of the Cauchy problem for (2.1). First we observe that the maximum principle of Section 2 holds for the Dirichlet boundary problem by a perturbation argument adding $\epsilon\omega^p$. Specifically, we have the following proposition:

PROPOSITION 11.1 *Let (M, g) be a Kähler manifold whose curvature operator $\mathrm{Rm} \in \mathcal{C}_p$. Let Ω be a bounded domain in M. Assume that $\phi(x,t)$ satisfies that*

$$\begin{cases} \frac{\partial}{\partial t}\phi(x,t) + \Delta_{\overline{\partial}}\phi(x,t) = 0, \\ \phi(x,t)|_{\partial\Omega} \ge 0, \\ \phi(x,0) = \phi_0(x) \ge 0. \end{cases} \tag{11.1}$$

Then $\phi(x,t) \ge 0$ for $t > 0$.

Note that the boundary condition $\phi(x,t)|_{\partial\Omega} \geq 0$ means that at any $x \in \partial\Omega$ and for any $v_1, \ldots, v_p$, $\phi(v_1, \ldots, v_p; \overline{v}_1, \ldots, \overline{v}_p) \geq 0$ at x. Namely, $\phi(x,t)|_{\partial\Omega}$ does not mean the (p, p)-form is restricted to the boundary.

This will help us to obtain a needed estimate to obtain a global solution in the case that M is a noncompact, complete manifold via some a priori estimates. A basic assumption is needed to ensure even the short-time existence of the Cauchy problem on open manifolds. Here we assume that there exists a positive constant a such that

$$\mathcal{B} \doteqdot \int_M |\phi_0(y)| \exp(-ar^2(y)) d\mu(y) < \infty, \tag{11.2}$$

where $r(x)$ is the distance function to some fixed point $o \in M$. The pointwise norm $|\cdot|$ for ϕ is defined as

$$|\phi|^2 = \frac{1}{(p!)^2} \sum \phi_{I_p, \overline{J}_p} \overline{\phi_{K_p, \overline{L}_p}} g^{i_1 \overline{k}_1} \cdots g^{i_p \overline{k}_p} g^{l_1 \overline{j}_1} \cdots g^{l_p \overline{j}_p}.$$

By basic linear algebra, for example, lemma 2.4 of [39], it is easy to see that for positive (p, p)-forms, there exists $C_{p,m}$ such that

$$|\phi|(x) \leq C_{p,m} |\Lambda\phi|(x). \tag{11.3}$$

Now the existence of the solution to the Cauchy problem can be proved for any continuous positive (p, p)-form $\phi_0(x)$ satisfying (11.2).

PROPOSITION 11.2 *Let (M, g) be a Kähler manifold whose curvature operator* $\mathrm{Rm} \in \mathcal{C}_p$. *Assume that $\phi_0(x)$ satisfies* (11.2); *then there exists T_0 such that the Cauchy problem*

$$\begin{cases} \frac{\partial}{\partial t}\phi(x,t) + \Delta_{\overline{\partial}}\phi(x,t) = 0, \\ \phi(x,0) = \phi_0(x) \geq 0, \end{cases} \tag{11.4}$$

has a solution $\phi(x,t)$ on $M \times [0, T_0]$. Moreover, $\phi(x,t) \geq 0$ on $M \times [0, T_0]$ and satisfies the estimate

$$|\phi|(x,t) \leq \mathcal{B} \cdot \frac{C(m,p)}{V_x(\sqrt{t})} \exp(2ar^2(x)). \tag{11.5}$$

Here $V_x(r)$ is the volume of ball $B(x,r)$.

PROOF. Let Ω_μ be a sequence of exhaustion bounded domains. By the standard theory on the linear parabolic system [14, 20], there exist solutions $\phi_\mu(x,t)$ on $\Omega_\mu \times [0, \infty)$ such that $\phi_\mu(x,t) = 0$ on $\partial\Omega_\mu \times [0, \infty)$. Note that in terms of the language of [25], $\phi_\mu(x,t) = 0$ on $\partial\Omega_\mu$ means that both the tangential part $\mathsf{t}\phi_\mu$ and the normal part $\mathsf{n}\phi_\mu$ vanish on $\partial\Omega_\mu$. Hence this is different from the more traditional *relative* and *absolute* boundary value problems for differential forms, which require $\mathsf{t}\phi = \mathsf{t}\delta\phi = 0$ and $\mathsf{n}\phi = \mathsf{n}d\phi = 0$, respectively. Nevertheless, it is a boundary condition (which was studied in [25]) such that together with $\Delta_{\overline{\partial}}\phi = 0$ it is hypo-elliptic and the Schauder estimate of [38] applies.

To get a global solution we shall prove that there exist uniform (in terms of μ) estimates so that we can extract a convergent subsequence. Note that $\Lambda^p \phi_\mu$ is a solution to the heat equation and $|\phi_\mu| \le C_{m,p} \Lambda^p \phi_\mu$. Let

$$u(x,t) = \int_M H(x,y,t)|\phi_0|(y)d\mu(y),$$

where $H(x,y,t)$ is the positive heat kernel of M. By the fundamental heat kernel estimate of Li and Yau [23], it is easy to see that, under assumption (11.2), there exists T_0 such that $u(x,t)$ is finite on $K \times [0, T_0]$ for any compact subset K. It is easy to see that $|\phi_\mu|(x,t) \le C_{m,p} \Lambda^p \phi_\mu \le C'_{p,m} u(x,t)$ by (11.3) and the maximum principle for the scalar heat equation. Now the interior Schauder estimates [38] (see also [26, theorem 6.2.6] for the corresponding estimates in the elliptic cases) imply that for any given $\alpha \in (0,1)$ and any compact subset $K \subset M$, for any μ with $K \subset \Omega_\mu$, ϕ_μ satisfies the estimate

$$\|\phi_\mu\|_{2,\alpha,\frac{\alpha}{2},K\times[0,T_0]} \le C(K, p, m, \|\phi_\mu\|_{\infty,K\times[0,T_0]}).$$

Here $\|\cdot\|_{2,\alpha,\alpha/2}$ is the $C^{2,\alpha}$-Hölder norm on the parabolic region. Since we estimate $\|\phi_\mu\|_{\infty,K\times[0,T_0]}$ by $u(x,t)$ uniformly, we have established the uniform estimates so that, after passing to a subsequence, $\{\phi_\mu(x,t)\}$ converges to a solution $\phi(x,t)$ on $M \times [0,T_0]$.

It is obvious from the construction that $\phi(x,t) \ge 0$. To prove the estimate (11.5), appealing to Li and Yau's upper estimate

$$H(x,y,t) \le \frac{C(n)}{V_x(\sqrt{t})} \exp\left(-\frac{r^2(x,y)}{5t}\right),$$

we can derive that for $0 \le t \le T_0 \le \frac{1}{10a}$,

$$\begin{aligned} u(x,t) &\le \frac{C(m)}{V_x(\sqrt{t})} \int_M \exp\left(-\frac{r^2(x,y)}{5t} + ar^2(y)\right)|\phi_0|(y) \exp(-ar^2(y))d\mu(y) \\ &\le \mathcal{B} \cdot \frac{C(m)}{V_x(\sqrt{t})} \exp(2ar^2(x)). \end{aligned}$$

In the second inequality above we used the estimate that for $0 \le t \le T_0 \le \frac{1}{10a}$

$$-\frac{r^2(x,y)}{5t} + ar^2(o,y) \le -\frac{r^2(x,y)}{5t} + 2ar^2(o,x) + 2ar^2(x,y) \le 2ar^2(x).$$

□

It is clear from the proof that if $\phi_0(x)$ satisfies the stronger assumption that

$$\int_M |\phi_0|(y) \exp(-ar^{2-\delta}(y))d\mu(y) < \infty \tag{11.6}$$

for some positive constants δ and a, then the Cauchy problem has a global solution on $M \times [0, \infty)$.

To deform a general (p, p)-form, we need the following generalization on a well-known lemma of Bishop-Goldberg concerning $(1, 1)$-forms on manifolds with $\mathcal{C}_1$. This also holds the key to extending Proposition 2.2 to noncompact manifolds.

LEMMA 11.3 *Assume that (M, g) satisfies $\mathcal{C}_2$. Then for any (p, q)-form ϕ,*

$$\langle \mathcal{KB}(\phi), \overline{\phi} \rangle \le 0. \tag{11.7}$$

PROOF. We shall check for the (p, p)-forms since the argument is the same for (p, q)-forms. For

$$\phi = \frac{1}{(p!)^2} \sum \phi_{I_p, \overline{J}_p} (\sqrt{-1} dz^{i_1} \wedge dz^{\overline{j}_1}) \wedge \cdots \wedge (\sqrt{-1} dz^{i_p} \wedge dz^{\overline{j}_p}),$$

where the summation is for $1 \le i_1, \ldots, i_p, j_1, \ldots, j_p \le m$. Under a normal coordinate,

$$\langle \phi, \overline{\psi} \rangle = \frac{1}{(p!)^2} \sum \phi_{I_p, \overline{J}_p} \overline{\psi_{I_p, \overline{J}_p}}.$$

Recall that also under the normal coordinate, $\langle \mathrm{Rm}(dz^k \wedge dz^{\overline{l}}), \overline{dz^s \wedge dz^{\overline{t}}} \rangle = R_{l\overline{k}s\overline{t}}$. It is easy to check that $\mathcal{C}_2$ implies that

$$\langle \mathrm{Rm}(Z_1^* \wedge \overline{W}_1^* + Z_2^* \wedge \overline{W}_2^*), \overline{Z_1^* \wedge \overline{W}_1^* + Z_2^* \wedge \overline{W}_2^*} \rangle \ge 0$$

for any $(1, 0)$-forms Z_1^*, Z_2^*, W_1^*, and W_2^*. We shall prove the claim by computing the expression under a normal coordinate. For any fixed I_p, J_p and μ, ν with $1 \le \mu, \nu \le p$, we can define

$$\overline{\eta}_\mu \doteqdot \sum_{k_\mu = 1}^{m} \phi_{i_1 \cdots (k_\mu)_\mu \cdots i_p, \overline{J}_p} dz^{\overline{k}_\mu}, \quad \xi_\nu \doteqdot \sum_{l_\nu = 1}^{m} \phi_{I_p, \overline{j}_1 \cdots \overline{(l_\nu)_\nu} \cdots \overline{j}_p} dz^{l_\nu}.$$

Now Rm being in $\mathcal{C}_2$ implies that

$$\langle \mathrm{Rm}(dz^{i_\mu} \wedge \overline{\eta}_\mu - \xi_\nu \wedge dz^{\overline{j}_\nu}), \overline{dz^{i_\mu} \wedge \overline{\eta}_\mu - \xi_\nu \wedge dz^{\overline{j}_\nu}} \rangle \ge 0. \tag{11.8}$$

Now using that

$$\begin{aligned} \mathrm{Rm}(dz^{i_\mu} \wedge \overline{\eta}_\mu - \xi_\nu \wedge dz^{\overline{j}_\nu}) = & \sum_{k_\mu, s t} \phi_{i_1 \cdots (k_\mu)_\mu \cdots i_p, \overline{J}_p} R_{k_\mu \overline{i}_\mu s \overline{t}} dz^s \wedge dz^{\overline{t}} \\ & - \sum_{l_\nu, s, t} \phi_{I_p, \overline{j}_1 \cdots \overline{(l_\nu)_\nu} \cdots \overline{j}_p} R_{j_\nu \overline{l}_\nu s \overline{t}} dz^s \wedge dz^{\overline{t}} \end{aligned}$$

we can expand the left-hand side of (11.8) and obtain that

$$
\begin{aligned}
0 \le & \sum_{k_\mu, k'_\mu} \phi_{i_1 \cdots (k_\mu)_\mu \cdots i_p, \overline{J}_p} R_{k_\mu \overline{i}_\mu i_\mu \overline{k'}_\mu} \overline{\phi_{i_1 \cdots (k'_\mu)_\mu \cdots i_p, \overline{J}_p}} \\
& - \sum_{l_\nu, k'_\mu} \phi_{I_p, \overline{j}_1 \cdots \overline{(l_\nu)_\nu} \cdots \overline{j}_p} R_{j_\nu \overline{l}_\nu i_\mu \overline{k'}_\mu} \overline{\phi_{i_1 \cdots (k'_\mu)_\mu \cdots i_p, \overline{J}_p}} \\
& - \sum_{k_\mu, l'_\nu} \phi_{i_1 \cdots (k_\mu)_\mu \cdots i_p, \overline{J}_p} R_{k_\mu \overline{i}_\mu l'_\nu \overline{j}_\nu} \overline{\phi_{I_p, \overline{j}_1 \cdots \overline{(l'_\nu)_\nu} \cdots \overline{j}_p}} \\
& + \sum_{l_\nu, l'_\nu} \phi_{I_p, \overline{j}_1 \cdots \overline{(l_\nu)_\nu} \cdots \overline{j}_p} R_{j_\nu \overline{l}_\nu l'_\nu \overline{j}_\nu} \overline{\phi_{I_p, \overline{j}_1 \cdots \overline{(l'_\nu)_\nu} \cdots \overline{j}_p}}.
\end{aligned} \tag{11.9}
$$

Now summing for all $1 \le i_1, \ldots, i_p, j_1, \ldots, j_p \le m$ and $1 \le \mu, \nu \le p$, a tedious, but straightforward checking shows that the total sum of the right-hand side above is $-2\langle \mathcal{KB}(\phi), \overline{\phi} \rangle$. □

An immediate consequence of the above lemma is that *any harmonic* (p, q)*-form on a compact Kähler manifold with* $\mathcal{C}_2$ *must be parallel*. This fact was known for harmonic $(p, 0)$-forms under the weaker assumption that Ric ≥ 0 and for harmonic $(1, 1)$-forms under the nonnegativity of bisectional curvature. In fact, using the full power of the uniformization result of Mori, Siu, Yau, and Mok, the result holds even under $\mathcal{C}_1$. Hence it does not give any new information for compact Kähler manifolds.

Another consequence of Lemma 11.3 is the following result, which generalizes lemma 2.1 of [35] by virtually the same argument.

COROLLARY 11.4 *Let* M^m *be a complete Kähler manifold with* $\mathcal{C}_2$. *Let* $\phi(x, t)$ *be a* (p, p)*-form satisfying* (2.1) *on* $M \times [0, T]$. *Then* $|\phi|(x, t)$ *is a subsolution of the heat equation.*

This together with the proof to Proposition 11.2 gives the following improvement on the existence of the Cauchy problem for initial (p, p)-forms not necessarily positive.

PROPOSITION 11.5 *Let* (M, g) *be a Kähler manifold whose curvature operator* Rm $\in \mathcal{C}_2$. *Assume that* $\phi_0(x)$ *satisfies* (11.2)*; then there exists* T_0 *such that the Cauchy problem* (11.4) *has a solution* $\phi(x, t)$ *on* $M \times [0, T_0]$. *Moreover,* (11.5) *holds.*

PROOF. Observe that $\phi_\mu(x, t)$ in the proof of Proposition 11.2 satisfies that $|\phi_\mu|(x, t)$ is a subsolution to the heat equation, hence $|\phi_\mu|(x, t) \le u(x, t)$. The rest of the proof of Proposition 11.2 applies here. □

A more important application of the lemma is the following extension of Proposition 2.2. This also extends theorem 2.1 of [35].

THEOREM 11.6 *Let (M, g) be a complete, noncompact Kähler manifold with $\mathcal{C}_p$. Let $\phi(x,t)$ be a (p,p)-form satisfying* (2.1) *on $M \times [0,T]$. Assume that $\phi(x,0) \geq 0$ and satisfies* (11.2). *Assume further that for some $a > 0$,*

$$\liminf_{r\to\infty} \int_0^T \int_{B_o(r)} |\phi|^2(x,t) \exp(-ar^2(x)) d\mu(x) dt < \infty. \tag{11.10}$$

Then $\phi(x,t) \geq 0$. Moreover, (11.5) *holds.*

Before we prove the theorem, we should remark that even though Proposition 11.2 provides a solution to the Cauchy problem that is a positive (p,p)-form, it is also useful to be able to assert that certain solutions, which are not constructed by Proposition 11.2, preserve the positivity. For example, if $\phi = \sqrt{-1}\,\partial\overline{\partial}\eta$ and η satisfies (2.1), then it is easy to see that ϕ satisfies (2.1) since $\Delta_{\overline{\partial}}$ is commutable with ∂ and $\overline{\partial}$. If we know that $\sqrt{-1}\,\partial\overline{\partial}\eta \geq 0$ at $t = 0$, it is desirable to know when we have $\phi(x,t) \geq 0$.

PROOF. We employ the localization technique of [35]. Let σ_R be a cutoff function between 0 and 1, being 1 in the annulus $A(R/4, 4R) = B(o,4R) \setminus B(o,R/4)$ and supported in the annulus $A(R/8, 8R)$. Let

$$u_R(x,t) = \int_M H(x,y,t)|\phi|(y,0)\sigma_R(y)d\mu(y),$$

$$u(x,t) = \int_M H(x,y,t)|\phi|(y,0)d\mu(y).$$

Clearly $u_R(x,t) \leq u(x,t)$. However, the following result is proved in [35, lemma 2.2].

LEMMA 11.7 (Ni-Tam) *Assume that $\phi(x,0)$ satisfies* (11.2). *Then there exists $T_0 > 0$, depending only on a, such that for $R \geq \max\{\sqrt{T_0}, 1\}$ the following are true*:

(i) *There exists a function $\tau = \tau(r)$ with $\lim_{r\to\infty} \tau(r) = 0$ such that for all $(x,t) \in A_o(R/2, 2R) \times [0, T_0]$,*

$$u(x,t) \leq u_R(x,t) + \tau(R).$$

(ii) *For any $r > 0$,*

$$\lim_{R\to\infty} \sup_{B_o(r)\times[0,T_0]} u_R = 0.$$

Lemma 11.3 above implies that $(u_R(x,t) + \tau(R))\omega^p$ can be used as a barrier on $\partial B_o(R) \times [0, T_0]$, since by Corollary 11.4 and the maximum principle on complete, noncompact Riemannian manifolds, that is where assumption (11.10) is needed,

$|\phi|(x,t) \le u(x,t) \le u_R(x,t) + \tau(R)$ on $\partial B_o(R) \times [0, T_0]$. In fact,

$$\big((u_R(x,t) + \tau(R))\omega^p + \phi\big)(v_1, \dots, v_p; \overline{v}_1, \dots, \overline{v}_p) \ge u_R(x,t) + \tau(R) - |\phi|(x,t) \ge 0 \tag{11.11}$$

for any $(v_1, \dots, v_p)$ that can be extended into a unitary basis of $T'_x M$. Now applying the argument of Proposition 2.2 to $B_0(R) \times [0, T]$, we can conclude that

$$(u_R(x,t) + \tau(R))\omega^p + \phi \ge 0$$

on $B_o(R) \times [0, T_0]$ as a (p, p)-form since $\phi(x, 0) \ge 0$. Now the result follows by letting $R \to \infty$ and the fact that $\lim_{R\to\infty} \sup_{B_o(r)\times[0,T_0]} u_R = 0$ proved in the lemma, and $\tau(R) \to 0$. Since $|\phi|(x,t) \le u(x,t)$, estimate (11.5) follows as before. □

We devote the rest of this section to the proofs of Theorems 4.1 and 6.1 for the case where M is noncompact complete. Since one can pick a small $\delta > 0$ and shift the time $t \to t - \delta$ and multiply the expression Q by $t - \delta$, we may assume without loss of generality that ϕ, $\partial^*\phi$, $\overline{\partial}^*\phi$, and $\overline{\partial}^*\partial^*\phi$ are all smooth up to $t = 0$.

First we observe that if ϕ is a positive (p, p)-form, then $\Lambda^p\phi(x,t)$ is a nonnegative solution to the heat equation and hence satisfies (11.10) by the estimate of Li and Yau. Specifically,

$$\Lambda^p\phi(x,t) \le \Lambda^p\phi(o,1) \cdot \frac{1}{t^m} \cdot \exp\left(\frac{r^2(x)}{4(1-t)}\right).$$

In particular, for $\frac{\delta}{2} < T < 1 - \delta$, one can find $a > 0$ such that

$$\int_M (\Lambda^p\phi)^2\left(x, \frac{\delta}{2}\right) \exp(-ar^2(x))d\mu(x) + \int_{\frac{\delta}{2}}^T \int_M (\Lambda^p\phi)^2 \exp(-ar^2(x))d\mu(x)dt < \infty.$$

Since $|\phi| \le C_{m,p}\Lambda^p\phi$ we can conclude that $|\phi|$ satisfies the above estimate.

Applying Lemma 11.3 to $(p-1, p)$- and $(p, p-1)$-forms implies the following estimates: There exists $c_{m,p} > 0$ such that

$$\left(\Delta - \frac{\partial}{\partial t}\right)|\phi|^2 \ge c_{m,p}(|\partial^*\phi|^2 + |\overline{\partial}^*\phi|^2), \tag{11.12}$$

$$\left(\Delta - \frac{\partial}{\partial t}\right)(|\partial^*\phi|^2 + |\overline{\partial}^*\phi|^2) \ge c_{m,p}|\overline{\partial}^*\partial^*\phi|^2, \tag{11.13}$$

$$\left(\Delta - \frac{\partial}{\partial t}\right)|\overline{\partial}^*\partial^*\phi|^2 \ge 0. \tag{11.14}$$

By the same argument as in the proof of lemma 1.4 in [30], we can conclude that there exists $a' > 0$ such that

$$\int_{\frac{\delta}{2}}^{T}\int_{M}(|\phi|^2+|\partial^*\phi|^2+|\bar\partial^*\phi|^2+|\bar\partial^*\partial^*\phi|^2)\exp(-a'r^2(x))<\infty. \tag{11.15}$$

Note that by the mean value theorem for the subsolution to the heat equation [22], one can obtain pointwise estimates for $|\phi|^2+|\partial^*\phi|^2+|\bar\partial^*\phi|^2+|\bar\partial^*\partial^*\phi|^2$ at $t=\delta$. Now with the help of the argument for the compact case, the same proof as used for Theorem 11.6 via the barrier argument, applied to Q viewed as a $(p-1,p-1)$-form-valued Hermitian symmetric tensor, proves Theorem 4.1 on complete, noncompact manifolds.

The corresponding result with the Kähler-Ricci flow, namely Theorem 6.1, is very similar. Hence we keep it brief. Due to the bound on the curvature, there exists a positive constant $\alpha_{m,A}$, depending on the upper bound A of the curvature tensor, so that

$$\left(\Delta-\frac{\partial}{\partial t}\right)(e^{-\alpha_{m,A}t}\cdot|\phi|^2)\geq c_{m,p}e^{-\alpha_{m,A}t}\cdot(|\partial^*\phi|^2+|\bar\partial^*\phi|^2), \tag{11.16}$$

$$\left(\Delta-\frac{\partial}{\partial t}\right)(e^{-\alpha_{m,A}t}\cdot(|\partial^*\phi|^2+|\bar\partial^*\phi|^2))\geq c_{m,p}e^{-\alpha_{m,A}t}\cdot|\bar\partial^*\partial^*\phi|^2, \tag{11.17}$$

$$\left(\Delta-\frac{\partial}{\partial t}\right)(e^{-\alpha_{m,A}t}\cdot|\bar\partial^*\partial^*\phi|^2)\geq 0. \tag{11.18}$$

There are modified pointwise estimates for the positive solutions coupled with the Ricci flow to replace the Li-Yau estimate. See, for example, [32, theorem 2.7] (a result of Guenther [15]). There is a corresponding mean value theorem for the nonnegative subsolutions to the heat equation. See, for example, [31, theorem 1.1]. Putting them together, a similar argument as made above applies to Theorem 6.1.

Remark 11.8. The argument here in fact proves theorem 1.1 of [30] without assumption (1.5) there, since that assumption is a consequence of the semipositivity of h and Li and Yau's estimate for positive solutions to the heat equation.

Appendix

Here we include Wilking's proof [41] to Theorem 3.1 (also included in [7]) following the notation of Section 3.

Recall that $\mathrm{ad}_v:\mathfrak{so}(n,\mathbb{C})\to\mathfrak{so}(n,\mathbb{C})$ mapping w to $[v,w]$. The operator $\mathrm{ad}_{(\cdot)}$ can be viewed as a map from $\mathfrak{so}(n,\mathbb{C})$ to the space of endomorphisms of $\mathfrak{so}(n,\mathbb{C})$. It is the derivative of Ad, the adjoint action of $\mathsf{SO}(n,\mathbb{C})$ on $\mathfrak{so}(n,\mathbb{C})$, which maps $\mathsf{SO}(n,\mathbb{C})$ to automorphisms of $\mathfrak{so}(n,\mathbb{C})$. This is a basic fact in Lie group theory. Another basic fact from Lie group theory asserts that $e^{\mathrm{ad}_v}=\mathrm{Ad}(\mathrm{Exp}(v))$.

For the proof of Wilking's theorem, we need to recall the following identity for $\mathrm{Rm}^{\#}$:

$$\langle \mathrm{Rm}^{\#}(v), w\rangle = \frac{1}{2}\sum_{\alpha,\beta}\langle [\mathrm{Rm}(b^{\alpha}), \mathrm{Rm}(b^{\beta})], v\rangle\langle [b^{\alpha}, b^{\beta}], w\rangle. \tag{A.1}$$

Here $\{b^{\alpha}\}$ is a basis for $\mathfrak{so}(n, \mathbb{C})$. To prove the invariance of $\mathcal{C}_{\Sigma}$, by Hamilton's tensor maximum principle, it suffices to show that if $\langle \mathrm{Rm}(v_0), \overline{v}_0\rangle = 0$, $\langle \mathrm{Rm}^{\#}(v_0), \overline{v}_0\rangle \geq 0$. Here we identify $\bigwedge^2(\mathbb{C}^n)$ with $\mathfrak{so}(n, \mathbb{C})$ and observe that the action of $\mathsf{SO}(n, \mathbb{C})$ on $\bigwedge^2(\mathbb{C}^n)$ is the same as the adjoint action under the identification.

Given the above basic facts from Lie group theory, for any $b \in \mathfrak{so}(n, \mathbb{C})$ and $z \in \mathbb{C}$, consider $(\mathrm{Ad}(\mathrm{Exp}(zb)))(v_0) = \mathrm{Exp}(zb)\cdot v_0 \cdot \mathrm{Exp}(-zb)$. Since $\mathrm{Exp}(zb) \in \mathsf{SO}(n, \mathbb{C})$, we conclude that $(\mathrm{Ad}(\mathrm{Exp}(zb)))(v_0) \in \Sigma$. Hence $v(z) = e^{z\ \mathrm{ad}_b}\cdot v_0 \in \Sigma$. Thus if we define

$$I(z) := \langle \mathrm{Rm}(v(z)), \overline{v(z)}\rangle$$

it is clear that $I(z) \geq 0$ and $I(0) = 0$, which implies that

$$\frac{\partial^2}{\partial z \partial \overline{z}} I(z)\bigg|_{z=0} \geq 0.$$

Hence for any $b \in \mathfrak{so}(n, \mathbb{C})$, $\langle \mathrm{Rm}(\mathrm{ad}_b(v_0)), \overline{\mathrm{ad}_b(v_0)}\rangle \geq 0$, which can be equivalently written as

$$\langle \mathrm{Rm}\cdot \mathrm{ad}_{v_0}(b), \mathrm{ad}_{\overline{v}_0}(\overline{b})\rangle \geq 0. \tag{A.2}$$

This is equivalent to $-\mathrm{ad}_{\overline{v}_0}\cdot \mathrm{Rm}\cdot \mathrm{ad}_{v_0} \geq 0$, as a Hermitian symmetric tensor.

By equation (A.1) $\langle \mathrm{Rm}^{\#}(v_0), \overline{v}_0\rangle \geq 0$ is the same as

$$\mathrm{trace}(-\mathrm{ad}_{\overline{v}_0}\cdot \mathrm{Rm}\cdot \mathrm{ad}_{v_0}\cdot \mathrm{Rm}) \geq 0.$$

This estimate can be derived from (A.2) as follows. Let λ_{α} be the eigenvalues of $-\mathrm{ad}_{\overline{v}_0}\cdot \mathrm{Rm}\cdot \mathrm{ad}_{v_0}$ with eigenvectors b^{α}. Then for $\lambda_{\alpha} > 0$, $b^{\alpha} = \frac{1}{\lambda_{\alpha}}\mathrm{ad}_{\overline{v}_0}(w^{\alpha})$, where $w^{\alpha} = \mathrm{Rm}\cdot \mathrm{ad}_{v_0}(b^{\alpha})$. At the same time

$$\begin{aligned}
\mathrm{trace}(-\mathrm{ad}_{\overline{v}_0}\cdot \mathrm{Rm}\cdot \mathrm{ad}_{v_0}\cdot \mathrm{Rm}) &= \sum \langle \mathrm{Rm}\cdot(-\mathrm{ad}_{\overline{v}_0}\cdot \mathrm{Rm}\cdot \mathrm{ad}_{v_0})(b^{\alpha}), \overline{b^{\alpha}}\rangle \\
&= \sum_{\lambda_{\alpha}>0} \lambda_{\alpha}\langle \mathrm{Rm}(b^{\alpha}), \overline{b^{\alpha}}\rangle \\
&= \sum_{\lambda_{\alpha}>0} \frac{1}{\lambda_{\alpha}}\langle \mathrm{Rm}\cdot \mathrm{ad}_{\overline{v}_0}(w^{\alpha}), \mathrm{ad}_{v_0}(\overline{w^{\alpha}})\rangle \\
&= \sum_{\lambda_{\alpha}>0} \frac{1}{\lambda_{\alpha}}\langle \mathrm{Rm}\cdot \mathrm{ad}_{v_0}(\overline{w^{\alpha}}), \mathrm{ad}_{\overline{v}_0}(w^{\alpha})\rangle.
\end{aligned}$$

The last expression is nonnegative by (A.2).

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