

MATH 20B lecture notes

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Based on *Single Variable Calculus: Early Transcendentals*
(4th edition), by Rogawski, Adams, and Franzosa,
and the UCSD MATH 20B supplement

Fall 2026

Math 20B: Calculus for Science and Engineering

- **Book:** *Single Variable Calculus: Early Transcendentals* (4th edition), Jon Rogawski, Colin Adams, and Robert Franzosa (Chapters 5–11).
- **Supplementary notes:** Math 20B Supplement (UCSD).
- **Webpage:** <https://mathweb.ucsd.edu/~lbuzaglo/teaching/20BF26/index.html>

Grading

Higher of two formulas:

- (1) 20/20/20/40 (HW / MT1 / MT2 / Final)
- (2) 20/20/60 (HW / max(MT1, MT2) / Final)

Course objectives

- Integral calculus of one variable and its applications, with exponential, logarithmic, hyperbolic, and trigonometric functions.
- Methods of integration.
- Polar coordinates in the plane.

Some topics we will cover

- Integration and the fundamental theorem of calculus (review), net change
- Applications of integration: area between curves, volume, volumes of revolution
- Polar coordinates, area in polar coordinates

- Complex numbers and complex exponentials
- Methods of integration:
 - substitution
 - integration by parts
 - trigonometric and hyperbolic integrals
 - trigonometric substitution
 - partial fractions
- Improper integrals
- Sequences and infinite series
- Convergence tests: absolute and conditional convergence, ratio and root tests
- Power series, Taylor polynomials, Taylor series

About these notes

These notes are based on the course textbook and the UCSD supplementary notes for MATH 20B. Sections 1–5 summarize §§5.1–5.5 of the textbook, which is review from MATH 20A and will be covered quickly in the first lecture. A more detailed treatment of this material is in the separate notes *MATH 20A review*.

1 Approximating and computing area

This section summarizes §5.1 of the textbook.

If an object moves with constant velocity v for time Δt , it travels distance $v\Delta t$, which is the area of a rectangle under the graph of the velocity. This remains true for varying velocity: *distance traveled is the area under the graph of velocity*. To compute areas under curved graphs, we approximate by rectangles and take a limit.

1.1 Riemann sum approximations

Let f be continuous on $[a, b]$. Divide $[a, b]$ into N subintervals of equal width $\Delta x = \frac{b-a}{N}$, with endpoints $x_j = a + j\Delta x$ for $j = 0, 1, \dots, N$ (so $x_0 = a$ and $x_N = b$). Recall the *summation notation* $\sum_{j=m}^n a_j = a_m + a_{m+1} + \dots + a_n$; sums are linear, i.e. $\sum(a_j + b_j) = \sum a_j + \sum b_j$ and $\sum C a_j = C \sum a_j$.

Definition 1.1. Using rectangles of width Δx whose heights are the values of f at the right endpoints, left endpoints, or midpoints of the subintervals, we obtain the *right-endpoint*, *left-endpoint*, and *midpoint approximations*

$$R_N = \Delta x \sum_{j=1}^N f(x_j), \quad L_N = \Delta x \sum_{j=0}^{N-1} f(x_j), \quad M_N = \Delta x \sum_{j=0}^{N-1} f\left(\frac{x_j + x_{j+1}}{2}\right).$$

Example 1.2. For $f(x) = x^{-1}$ on $[1, 3]$ with $N = 4$, we have $\Delta x = \frac{1}{2}$ (Figure 1), and

$$\begin{aligned} R_4 &= \frac{1}{2} \left(f\left(\frac{3}{2}\right) + f(2) + f\left(\frac{5}{2}\right) + f(3) \right) = 0.95, \\ L_4 &= \frac{1}{2} \left(f(1) + f\left(\frac{3}{2}\right) + f(2) + f\left(\frac{5}{2}\right) \right) \approx 1.283, \\ M_4 &= \frac{1}{2} \left(f\left(\frac{5}{4}\right) + f\left(\frac{7}{4}\right) + f\left(\frac{9}{4}\right) + f\left(\frac{11}{4}\right) \right) \approx 1.090. \end{aligned}$$

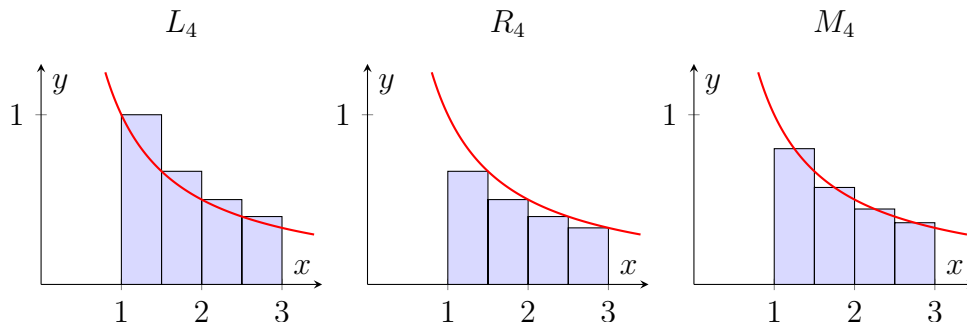


Figure 1: Left-endpoint, right-endpoint, and midpoint rectangles for $f(x) = x^{-1}$ on $[1, 3]$.

Remark 1.3. If f is increasing, then $L_N \leq \text{area} \leq R_N$; if f is decreasing, then $R_N \leq \text{area} \leq L_N$. In Example 1.2 the exact area turns out to be $\ln 3 \approx 1.099$, and M_4 is the best estimate.

1.2 Area as a limit

Using more, thinner rectangles makes the approximations better, and in the limit they become exact.

Theorem 1.4. *If f is continuous on $[a, b]$, then $\lim_{N \rightarrow \infty} R_N = \lim_{N \rightarrow \infty} L_N = \lim_{N \rightarrow \infty} M_N$. If moreover $f(x) \geq 0$ on $[a, b]$, this common limit is defined to be the area under the graph of f over $[a, b]$.*

For polynomials, these limits can be computed with the *power sums*

$$\sum_{j=1}^N j = \frac{N^2}{2} + \frac{N}{2}, \quad \sum_{j=1}^N j^2 = \frac{N^3}{3} + \frac{N^2}{2} + \frac{N}{6}, \quad \sum_{j=1}^N j^3 = \frac{N^4}{4} + \frac{N^3}{2} + \frac{N^2}{4}.$$

Example 1.5. Find the area under $f(x) = x^2 + 2x$ over $[1, 3]$.

Solution. Here $\Delta x = \frac{2}{N}$ and $f\left(1 + \frac{2j}{N}\right) = \frac{4}{N^2}j^2 + \frac{8}{N}j + 3$. By linearity and the power sums,

$$R_N = \frac{8}{N^3} \sum_{j=1}^N j^2 + \frac{16}{N^2} \sum_{j=1}^N j + \frac{6}{N} \sum_{j=1}^N 1 = \frac{50}{3} + \frac{12}{N} + \frac{4}{3N^2} \xrightarrow{N \rightarrow \infty} \frac{50}{3}.$$

For most other functions (e.g. $\cos x$ on $[0, \frac{\pi}{2}]$) such limits are hard to evaluate directly; the fundamental theorem of calculus (Section 4) will solve this problem.

2 The definite integral

This section summarizes §5.2 of the textbook.

2.1 Riemann sums and the definite integral

We now allow rectangles of different widths, with heights given by any value of f in each subinterval, possibly negative.

Definition 2.1. A *partition* P of $[a, b]$ of size N is a choice of points $a = x_0 < x_1 < \cdots < x_N = b$. Write $\Delta x_i = x_i - x_{i-1}$, and let the *norm* $\|P\|$ be the largest of the Δx_i . Given *sample points* $C = \{c_1, \dots, c_N\}$ with c_i in $[x_{i-1}, x_i]$, the *Riemann sum* is

$$R(f, P, C) = \sum_{i=1}^N f(c_i) \Delta x_i.$$

Each term $f(c_i) \Delta x_i$ is the *signed area* of a rectangle: positive if $f(c_i) > 0$ and negative if $f(c_i) < 0$. The approximations R_N , L_N , M_N are special Riemann sums.

Definition 2.2. A function f is *integrable* over $[a, b]$ if all Riemann sums approach the same limit as $\|P\| \rightarrow 0$, however the partitions and sample points are chosen. This limit is the *definite integral*

$$\int_a^b f(x) dx = \lim_{\|P\| \rightarrow 0} \sum_{i=1}^N f(c_i) \Delta x_i.$$

The function $f(x)$ is the *integrand*, and a, b are the *limits of integration*. The variable of integration is a dummy variable: $\int_a^b f(x) dx = \int_a^b f(t) dt$.

Theorem 2.3. If f is continuous on $[a, b]$, or continuous except at finitely many jump discontinuities, then f is integrable over $[a, b]$.

2.2 Signed area

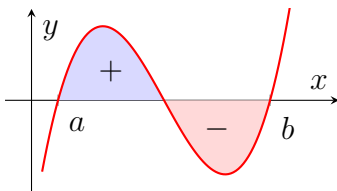


Figure 2: Signed area: the area above the x -axis counts positively and the area below counts negatively.

If f is integrable on $[a, b]$, then $\int_a^b f(x) dx$ is the *signed area* between the graph of f and the x -axis over $[a, b]$: area above the axis counts positively and area below counts negatively (Figure 2). In particular, if $f(x) \geq 0$, the integral is the area under the graph.

Example 2.4. The graph of $y = x - 2$ over $[0, 6]$ forms a triangle of area 2 below the axis (over $[0, 2]$) and a triangle of area 8 above it (over $[2, 6]$). Hence $\int_0^6 (x - 2) dx = -2 + 8 = 6$, whereas $\int_0^6 |x - 2| dx = 2 + 8 = 10$.

2.3 Properties of the definite integral

Definition 2.5. For $a < b$ we set $\int_b^a f(x) \, dx = -\int_a^b f(x) \, dx$, and we set $\int_a^a f(x) \, dx = 0$.

Theorem 2.6 (Properties of the definite integral). *Let f and g be integrable, and let C , K be constants.*

(1) **Constants:** $\int_a^b K \, dx = K(b - a)$.

(2) **Linearity:** $\int_a^b (f(x) + g(x)) \, dx = \int_a^b f(x) \, dx + \int_a^b g(x) \, dx$ and $\int_a^b C f(x) \, dx = C \int_a^b f(x) \, dx$.

(3) **Additivity:** $\int_a^c f(x) \, dx = \int_a^b f(x) \, dx + \int_b^c f(x) \, dx$ (for $a \leq b \leq c$, and in fact for all a, b, c).

(4) **Comparison:** if $g(x) \leq f(x)$ on $[a, b]$, then $\int_a^b g(x) \, dx \leq \int_a^b f(x) \, dx$.

(5) **Bounds:** if $m \leq f(x) \leq M$ on $[a, b]$, then $m(b - a) \leq \int_a^b f(x) \, dx \leq M(b - a)$.

Idea. Each property holds for Riemann sums (e.g. sums are linear, and $g(c_i)\Delta x_i \leq f(c_i)\Delta x_i$), and hence for their limits. Property (5) is (4) combined with (1).

Example 2.7. Since $\frac{1}{1+x^2}$ is decreasing on $[0, 1]$, it lies between $m = f(1) = \frac{1}{2}$ and $M = f(0) = 1$, so $\frac{1}{2} \leq \int_0^1 \frac{dx}{1+x^2} \leq 1$.

3 The indefinite integral

This section summarizes §5.3 of the textbook.

Definition 3.1. A function F is an *antiderivative* of f on an open interval (a, b) if $F'(x) = f(x)$ for all x in (a, b) .

Theorem 3.2 (The general antiderivative). *If F is an antiderivative of f on (a, b) , then every antiderivative of f on (a, b) has the form $F(x) + C$ for some constant C .*

Idea. If $F' = G' = f$, then $(G - F)' = 0$, so $G - F$ is constant by the mean value theorem. Graphically, the antiderivatives of f are the vertical shifts of the graph of F , which all have the same tangent slopes.

Notation 3.3. We write $\int f(x) \, dx = F(x) + C$ to mean $F'(x) = f(x)$, and call $F(x) + C$ the *general antiderivative* or *indefinite integral* of f . The constant C is the *constant of integration*.

Reading derivative formulas backwards gives the basic integrals below; each can be checked by differentiating the right-hand side.

Proposition 3.4 (Basic indefinite integrals).

$$\begin{aligned}\int x^n dx &= \frac{x^{n+1}}{n+1} + C \quad (n \neq -1), & \int \frac{dx}{x} &= \ln|x| + C, \\ \int e^{kx} dx &= \frac{1}{k} e^{kx} + C \quad (k \neq 0), & \int \sin x dx &= -\cos x + C, \\ \int \cos x dx &= \sin x + C, & \int \sec^2 x dx &= \tan x + C, \\ \int \csc^2 x dx &= -\cot x + C, & \int \sec x \tan x dx &= \sec x + C, \\ \int \csc x \cot x dx &= -\csc x + C.\end{aligned}$$

Moreover, indefinite integrals are linear: $\int (f(x) + g(x)) dx = \int f(x) dx + \int g(x) dx$ and $\int cf(x) dx = c \int f(x) dx$.

Remark 3.5. The formula $\int \frac{dx}{x} = \ln|x| + C$ holds on the whole domain $x \neq 0$: for $x > 0$ it is $\frac{d}{dx} \ln x = \frac{1}{x}$, and $\ln|x|$ is the even extension to $x < 0$, which also has derivative $\frac{1}{x}$.

Example 3.6. Integrating term by term,

$$\int \left(4x^3 - 6\sqrt{x} + \frac{2}{x} + e^{-3x} \right) dx = x^4 - 4x^{3/2} + 2 \ln|x| - \frac{1}{3} e^{-3x} + C.$$

One constant of integration suffices. Differentiating the answer recovers the integrand.

Caution. There are no product, quotient, or chain rules for integrals; rewrite the integrand algebraically first (e.g. $\frac{2x^3-7}{x^4} = \frac{2}{x} - 7x^{-4}$). The product and chain rules for derivatives will lead to integration by parts (§7.1) and substitution (§5.7).

Remark 3.7. The definite integral $\int_a^b f(x) dx$ is a *number* (a limit of Riemann sums), while the indefinite integral $\int f(x) dx$ is a *family of functions*. The fundamental theorem of calculus connects the two.

3.1 Initial value problems

An antiderivative of f is a solution of the *differential equation* $\frac{dy}{dx} = f(x)$. Its solutions are all $y = F(x) + C$, and an *initial condition* $y(x_0) = y_0$ selects one of them.

Definition 3.8. A differential equation together with an initial condition is called an *initial value problem*.

Example 3.9. A train moving at 30 m/s brakes with constant acceleration -5 m/s^2 from $t = 0$. How far does it travel before stopping?

Solution. Since $v(t) = \int (-5) dt = -5t + C$ and $v(0) = 30$, we get $v(t) = 30 - 5t$. Then $s(t) = \int v(t) dt = 30t - \frac{5}{2}t^2 + C_1$, and taking $s(0) = 0$ gives $C_1 = 0$. The train stops when $v(t) = 0$, i.e. $t = 6 \text{ s}$, having traveled $s(6) = 90 \text{ m}$.

4 The fundamental theorem of calculus, part I

This section summarizes §5.4 of the textbook.

The fundamental theorem of calculus (FTC) links differentiation and integration. Part I lets us compute definite integrals using antiderivatives, with no limits of Riemann sums.

Theorem 4.1 (The fundamental theorem of calculus, part I). *Assume that $a < b$ and that f is continuous on $[a, b]$. If F is an antiderivative of f on $[a, b]$, then*

$$\int_a^b f(x) \, dx = F(b) - F(a).$$

Idea. For any partition, the total change $F(b) - F(a)$ is the sum of the small changes $F(x_i) - F(x_{i-1})$. By the mean value theorem, each small change equals $f(c_i^*)\Delta x_i$ for some c_i^* in $[x_{i-1}, x_i]$, which is the signed area of a thin rectangle under the graph of f . So $F(b) - F(a)$ is a Riemann sum for f , and letting $\|P\| \rightarrow 0$ gives $\int_a^b f(x) \, dx$.

Notation 4.2. We write $F(x)|_a^b = F(b) - F(a)$.

Method (Evaluating $\int_a^b f(x) \, dx$).

(1) Find an antiderivative: $\int f(x) \, dx = F(x) + C$.

(2) Evaluate and subtract: $\int_a^b f(x) \, dx = F(x)|_a^b = F(b) - F(a)$.

Any antiderivative works, since the constant C cancels in $F(b) - F(a)$. The theorem also holds without assuming $a < b$.

Example 4.3.

(a) $\int_1^4 (x^{-1/2} + 2x^{3/2}) \, dx = \left(2x^{1/2} + \frac{4}{5}x^{5/2} \right) \Big|_1^4 = 29.6 - 2.8 = 26.8.$

(b) $\int_{-\pi/2}^{\pi/2} \cos x \, dx = \sin x \Big|_{-\pi/2}^{\pi/2} = 2$, so one hump of the cosine graph has area 2. Over $[-\frac{\pi}{2}, \frac{3\pi}{2}]$ the integral is $\sin \frac{3\pi}{2} - \sin(-\frac{\pi}{2}) = 0$: the hump below the axis cancels the hump above.

(c) $\int_0^{\pi/2} \cos x \, dx = \sin x \Big|_0^{\pi/2} = 1$, an area that was hard to compute as a limit of R_N in Section 1.

Two consequences worth remembering:

- For $n \geq 0$, $\int_0^b x^n \, dx = \frac{b^{n+1}}{n+1}$.
- If a and b are both positive or both negative, $\int_a^b \frac{dx}{x} = \ln |b| - \ln |a| = \ln \frac{b}{a}$. For example, $\int_2^{10} \frac{dx}{x} = \ln 5$, $\int_{-6}^{-2} \frac{dx}{x} = \ln \frac{1}{3}$, and $\int_1^e \frac{dx}{x} = 1$. (If $a < 0 < b$, FTC I does not apply, since $\frac{1}{x}$ is not continuous on $[a, b]$.)

Remark 4.4. FTC I is only useful if we can find a formula for an antiderivative. For $\int_0^2 \sqrt{1+x^3} dx$ there is no such formula, but Riemann sums still approximate the integral (e.g. $L_{1000} \approx 3.239$).

5 The fundamental theorem of calculus, part II

This section summarizes §5.5 of the textbook.

Part II goes the other way: it builds antiderivatives out of definite integrals, showing that integration and differentiation are inverse processes.

Definition 5.1. The *area function* of f with lower limit a is

$$A(x) = \int_a^x f(t) dt = \text{signed area from } a \text{ to } x.$$

For example, if $f(x) = 3x^2 + 2$ and $a = 1$, then $A(x) = (t^3 + 2t)|_1^x = x^3 + 2x - 3$, and $A'(x) = 3x^2 + 2 = f(x)$. This is no coincidence.

Theorem 5.2 (The fundamental theorem of calculus, part II). *Assume that f is continuous on an open interval I , and let a be in I . Then $A(x) = \int_a^x f(t) dt$ is an antiderivative of f on I ; that is,*

$$\frac{d}{dx} \int_a^x f(t) dt = f(x).$$

Idea. By additivity, $A(x+h) - A(x) = \int_x^{x+h} f(t) dt$ is the area of a thin sliver of width h (Figure 3). For small h its height is close to $f(x)$, so $A(x+h) - A(x) \approx hf(x)$. Dividing by h and letting $h \rightarrow 0$ (using continuity of f and the squeeze theorem) gives $A'(x) = f(x)$.

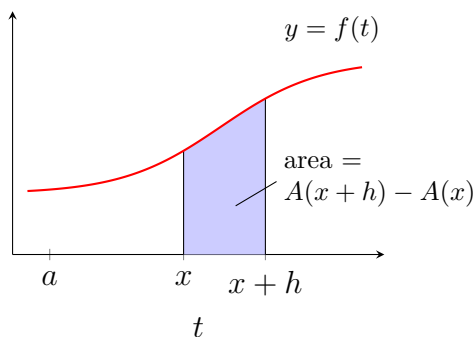


Figure 3: The change in the area function over $[x, x+h]$ is the area of a thin sliver, approximately $hf(x)$.

Remark 5.3 (Integration and differentiation are inverse operations).

$$f(x) \xrightarrow{\text{integrate}} \int_a^x f(t) dt \xrightarrow{\text{differentiate}} f(x) \quad (\text{FTC II}),$$

$$f(x) \xrightarrow{\text{differentiate}} f'(x) \xrightarrow{\text{integrate}} \int_a^x f'(t) dt = f(x) - f(a) \quad (\text{FTC I}).$$

In particular, *every continuous function on an open interval has an antiderivative*, even when no formula for one exists. For instance, $\int_0^x \sqrt{1+t^3} dt$ is an antiderivative of $\sqrt{1+x^3}$, although there is no elementary function with this derivative. Many important functions are defined this way, such as the Gauss error function $\text{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt$. Such functions can still be studied through their derivatives and numerical approximations.

Example 5.4. Let $A(x) = \int_1^x \sqrt{t^4 + 1} dt$. Then $A(1) = 0$, while $A(2)$ has no simple exact formula and must be approximated (R_{100} gives $A(2) \approx 2.58$). By FTC II, $A'(x) = \sqrt{x^4 + 1}$, so $A'(1) = \sqrt{2}$ and $A'(2) = \sqrt{17}$.

When the upper limit is a function of x , combine FTC II with the chain rule.

Example 5.5. Let $G(x) = \int_1^{x^3} \cos t dt$. Then $G(x) = A(x^3)$ with $A(x) = \int_1^x \cos t dt$ and $A'(x) = \cos x$, so

$$G'(x) = A'(x^3) \cdot 3x^2 = 3x^2 \cos(x^3).$$

Remark 5.6 (Reading A from the graph of f). Since $A' = f$ (Figure 4):

- where $f(x) > 0$, positive area is being added, so A is increasing; where $f(x) < 0$, A is decreasing;
- A has a local maximum where f changes sign from $+$ to $-$, and a local minimum where f changes sign from $-$ to $+$;
- the larger $|f(x)|$ is, the faster A changes.

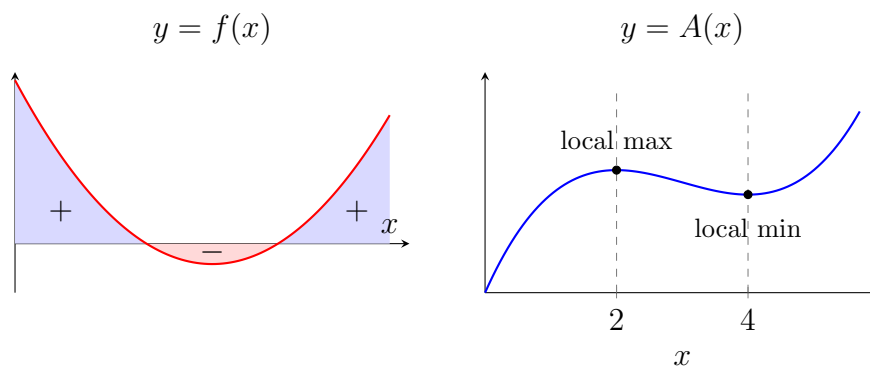


Figure 4: The sign of $f(x)$ determines where $A(x) = \int_0^x f(t) dt$ is increasing or decreasing.

6 Net change as the integral of a rate of change

This section covers §5.6 of the textbook.

So far we have used the definite integral to compute area. Its real usefulness comes from a second reading of the same formula: *the integral of a rate of change is the net change*.

Suppose water flows into an empty bucket at a rate of $r(t)$ liters per second, and let $s(t)$ be the quantity of water in the bucket at time t . If the flow rate were constant, say 1.5 L/s, the quantity after 4 seconds would be a product:

$$\text{quantity of water} = \text{flow rate} \times \text{time elapsed} = (1.5)4 = 6 \text{ L.}$$

When the flow rate varies (Figure 5), the product becomes an integral. Indeed $s'(t) = r(t)$, since $r(t)$ is by definition the rate at which the quantity of water is changing, and $s(0) = 0$ because the bucket starts empty. So FTC I gives

$$\underbrace{\int_0^4 r(t) dt}_{\text{area under the graph of the flow rate}} = s(4) - s(0) = \underbrace{s(4)}_{\text{water in the bucket at } t=4}$$

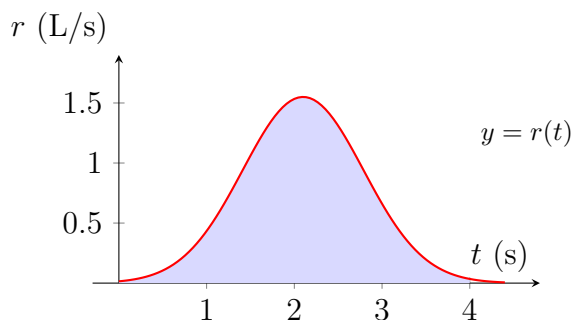


Figure 5: The quantity of water in the bucket at time $t = 4$ is the area under the graph of the flow rate $r(t)$ over $[0, 4]$.

More generally, $s(t_2) - s(t_1)$ is called the *net change* in $s(t)$ over the interval $[t_1, t_2]$, and FTC I expresses it as an integral.

Theorem 6.1 (Net change as the integral of a rate of change). *The net change in $s(t)$ over an interval $[t_1, t_2]$ is given by the integral*

$$\underbrace{\int_{t_1}^{t_2} s'(t) dt}_{\text{integral of the rate of change}} = \underbrace{s(t_2) - s(t_1)}_{\text{net change over } [t_1, t_2]}$$

Remark 6.2. In Theorem 6.1 the variable t does not have to be a time variable.

Example 6.3. Water leaks from a tank at a rate of $3 + 4t$ L/hour, where t is the number of hours after 8 AM. How much water is lost between 10 AM and 1 PM?

Solution. Let $s(t)$ be the quantity of water in the tank at time t . Since $3 + 4t$ is the rate at which water *leaves* the tank, the rate of change of the water *in* the tank is $s'(t) = -(3 + 4t)$. The times 10 AM and 1 PM correspond to $t = 2$ and $t = 5$, so the net change in $s(t)$ between them is

$$s(5) - s(2) = \int_2^5 s'(t) dt = - \int_2^5 (3 + 4t) dt = - (3t + 2t^2) \Big|_2^5 = -(65 - 14) = -51 \text{ liters.}$$

The tank lost 51 L between 10 AM and 1 PM.

An integral can also be estimated from numerical data, using the endpoint approximations of Section 1. Averaging L_N and R_N is usually more accurate than either one alone; this is the *trapezoidal approximation* (see §7.8 of the textbook).

Example 6.4 (Traffic flow). The number of cars per hour passing an observation point on a highway is the *traffic flow rate* $q(t)$ (in cars per hour).

- (a) Which quantity is represented by $\int_{t_1}^{t_2} q(t) dt$?
- (b) The flow rate past an observation point is recorded at 15-minute intervals between 7:00 and 9:00 AM, as follows.

t	7:00	7:15	7:30	7:45	8:00	8:15	8:30	8:45	9:00
$q(t)$	820	1150	1420	1680	1540	1290	980	720	560

Estimate the number of cars passing the point during this two-hour period.

Solution.

- (a) The flow rate is the rate of change of the number of cars that have passed the observation point, so by Theorem 6.1 the integral $\int_{t_1}^{t_2} q(t) dt$ is the total number of cars that passed the point during $[t_1, t_2]$.
- (b) The data are spaced at intervals of $\Delta t = 0.25$ h, so

$$L_N = 0.25(820 + 1150 + 1420 + 1680 + 1540 + 1290 + 980 + 720) = 2400,$$

$$R_N = 0.25(1150 + 1420 + 1680 + 1540 + 1290 + 980 + 720 + 560) = 2335,$$

where L_N uses the values of $q(t)$ at the left endpoints 7:00, 7:15, $\dots$, 8:45 and R_N the values at the right endpoints 7:15, 7:30, $\dots$, 9:00. Averaging,

$$\int_7^9 q(t) dt \approx \frac{1}{2}(L_N + R_N) = \frac{1}{2}(2400 + 2335) \approx 2370 \text{ cars.}$$

6.1 The integral of velocity

Let $s(t)$ be the position at time t of an object in linear motion. Its velocity is $v(t) = s'(t)$, so by Theorem 6.1 the integral of v is the net change in position, called the *displacement*:

$$\int_{t_1}^{t_2} v(t) dt = \int_{t_1}^{t_2} s'(t) dt = \underbrace{s(t_2) - s(t_1)}_{\text{displacement}}.$$

Caution. Displacement is not the same as *distance traveled*. If you walk 10 km and then return to your starting point, your displacement is 0 but the distance you traveled is 20 km. To compute distance traveled we integrate the *speed* $|v(t)|$, in which no cancellation can occur.

Theorem 6.5 (The integral of velocity). *For an object in linear motion with velocity $v(t)$,*

$$\text{displacement during } [t_1, t_2] = \int_{t_1}^{t_2} v(t) \, dt,$$

$$\text{distance traveled during } [t_1, t_2] = \int_{t_1}^{t_2} |v(t)| \, dt.$$

Example 6.6. A particle has velocity $v(t) = 6t - t^2$ m/s. Compute its displacement and the total distance it travels over $[0, 8]$, and indicate its trajectory with a motion diagram.

Solution. First, $\int v(t) \, dt = 3t^2 - \frac{1}{3}t^3 + C$. The displacement over $[0, 8]$ is

$$\int_0^8 v(t) \, dt = \left(3t^2 - \frac{1}{3}t^3\right)\Big|_0^8 = 192 - \frac{512}{3} = \frac{64}{3} \text{ m.}$$

For the distance traveled, note that $v(t) = t(6 - t)$ changes sign at $t = 6$: it is positive on $[0, 6]$ and negative on $[6, 8]$, as we see in Figure 6. Hence

$$\int_0^8 |v(t)| \, dt = \int_0^6 v(t) \, dt - \int_6^8 v(t) \, dt,$$

and we evaluate the two integrals separately:

$$[0, 6] : \quad \int_0^6 v(t) \, dt = \left(3t^2 - \frac{1}{3}t^3\right)\Big|_0^6 = 36 \text{ m,}$$

$$[6, 8] : \quad \int_6^8 v(t) \, dt = \left(3t^2 - \frac{1}{3}t^3\right)\Big|_6^8 = \frac{64}{3} - 36 = -\frac{44}{3} \text{ m.}$$

The total distance traveled is $36 + \frac{44}{3} = \frac{152}{3} = 50\frac{2}{3}$ m. Figure 7 is a motion diagram of the trajectory: the particle travels 36 m during the first 6 s and then backtracks $\frac{44}{3}$ m over the next 2 s, ending at $\frac{64}{3}$ m.

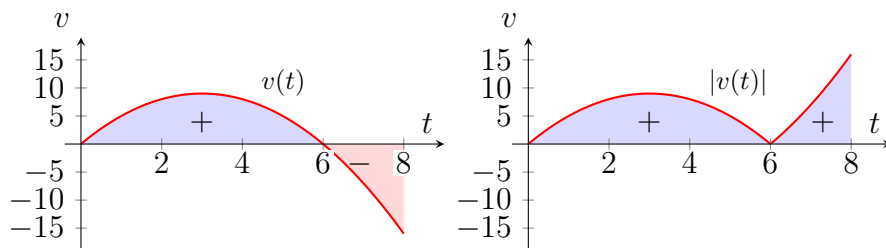


Figure 6: Graphs of $v(t) = 6t - t^2$ (left) and of the speed $|v(t)|$ (right). The displacement is the signed area; the distance traveled is the total area.

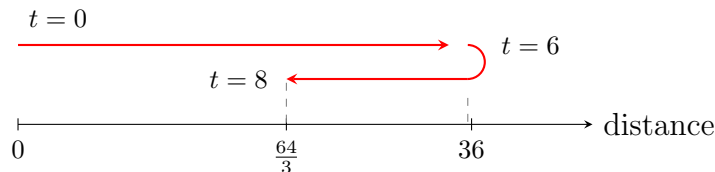


Figure 7: Motion diagram: the path of the particle along a straight line.

6.2 Total versus marginal cost

Definition 6.7. Let $C(x)$ be a manufacturer's cost to produce x units of a particular product or commodity. The derivative $C'(x)$ is called the *marginal cost*.

By Theorem 6.1, the cost of increasing production from a units to b units is the net change $C(b) - C(a)$, which is the integral of the marginal cost:

$$\text{cost of increasing production from } a \text{ units to } b \text{ units} = \int_a^b C'(x) \, dx.$$

Remark 6.8. In §3.4 of the textbook, the marginal cost at production level x_0 was defined as the cost $C(x_0 + 1) - C(x_0)$ of producing one additional unit. When x_0 is large compared to 1, this is approximated well by the derivative $C'(x_0)$, which is why economists refer to $C'(x)$ itself as the marginal cost.

Example 6.9. The marginal cost of producing x computer chips (in units of 1000) is $C'(x) = 60x^2 - 600x + 5000$ dollars per thousand chips.

- Find the cost of increasing production from 5000 to 8000 chips.
- Determine the total cost of producing 8000 chips, assuming that it costs \$2000 to set up the manufacturing run, i.e. $C(0) = 2000$.

Solution.

- The cost of increasing production from 5000 ($x = 5$) to 8000 ($x = 8$) chips is

$$\begin{aligned} C(8) - C(5) &= \int_5^8 (60x^2 - 600x + 5000) \, dx = (20x^3 - 300x^2 + 5000x) \Big|_5^8 \\ &= 31,040 - 20,000 = \$11,040. \end{aligned}$$

- The cost of increasing production from 0 to 8000 chips is

$$C(8) - C(0) = (20x^3 - 300x^2 + 5000x) \Big|_0^8 = \$31,040,$$

so the total cost of producing 8000 chips, including the set-up cost, is

$$C(8) = C(0) + 31,040 = 2000 + 31,040 = \$33,040.$$

7 The substitution method

This section covers §5.7 of the textbook.

Integration is generally harder than differentiation: there is no sure-fire method, and many antiderivatives cannot be expressed in terms of elementary functions. There are, however, a few general techniques. The first is the *substitution method*, which is essentially an inverse of the chain rule.

Remark 7.1. The word “integration” is used in two ways: for the process of computing a definite integral, and for the process of finding an antiderivative, that is, of evaluating an indefinite integral.

Consider $\int 2x \sec^2(x^2) dx$. We can evaluate it if we remember the chain rule calculation $\frac{d}{dx} \tan(x^2) = 2x \sec^2(x^2)$, which says that $\tan(x^2)$ is an antiderivative of $2x \sec^2(x^2)$, and therefore

$$\int \underbrace{2x}_{\substack{\text{derivative of} \\ \text{inside function}}} \sec^2 \left(\overbrace{x^2}^{\substack{\text{inside} \\ \text{function}}} \right) dx = \tan(x^2) + C.$$

A similar chain rule calculation shows that

$$\int \underbrace{(2x + 5)}_{\substack{\text{derivative of} \\ \text{inside function}}} \cos \left(\overbrace{x^2 + 5x}^{\substack{\text{inside} \\ \text{function}}} \right) dx = \sin(x^2 + 5x) + C.$$

Remark 7.2. Recall that a composite function has the form $f(g(x))$. For convenience we call $u = g(x)$ the *inside function* and $f(u)$ the *outside function*.

In both cases the integrand is the product of a composite function and the derivative of the inside function. The chain rule does not help if the derivative of the inside function is missing: we cannot use it to compute $\int \cos(x^2 + 5x) dx$, because the factor $2x + 5$ does not appear.

In general, if $F'(u) = f(u)$, then the chain rule gives $\frac{d}{dx} F(u(x)) = F'(u(x))u'(x) = f(u(x))u'(x)$, which translates into the following integration formula.

Theorem 7.3 (The substitution method). *If $F'(x) = f(x)$, and u is a differentiable function whose range includes the domain of f , then*

$$\int f(u(x))u'(x) dx = F(u(x)) + C.$$

7.1 Substitution using differentials

In practice we carry out substitutions using differentials. The symbols du and dx occurring in integrals may be treated as differentials, related as in §4.1 of the textbook by $du = \frac{du}{dx} dx$, and so

$$du = u'(x) dx. \tag{7.1}$$

For example, if $u = x^2$ then $du = 2x \, dx$, and if $u = \cos(x^3)$ then $du = -3x^2 \sin(x^3) \, dx$. When the integrand has the form $f(u(x))u'(x)$, we may use (7.1) to rewrite the entire integral, the dx term included, in terms of u and du :

$$\int \underbrace{f(u(x))}_{f(u)} \underbrace{u'(x) \, dx}_{du} = \int f(u) \, du.$$

This is the *change of variables formula*. It transforms an integral in the variable x into a hopefully simpler integral in the new variable u .

Remark 7.4. This symbolic calculus of substitution using differentials was invented by Leibniz and is considered one of his most important achievements. It reduces the otherwise complicated process of transforming integrals to a convenient procedure.

Method (Substitution).

- (1) Choose u and compute du .
- (2) Rewrite the integral in terms of u and du , and evaluate it.
- (3) Express the final answer in terms of x .

The key step is choosing the expression to assign to u ; often u can be chosen to be an inside function in a composition.

Example 7.5. Evaluate $\int 4x^3 \cos(x^4) \, dx$.

Solution. The integrand contains the composite function $\cos(x^4)$, so we set $u = x^4$. The differential $du = 4x^3 \, dx$ also appears, so we can carry out the substitution:

$$\int \underbrace{\cos(x^4)}_{\cos u} \underbrace{4x^3 \, dx}_{du} = \int \cos u \, du = \sin u + C = \sin(x^4) + C.$$

We check by differentiating: $\frac{d}{dx} \sin(x^4) = \cos(x^4) \cdot 4x^3$.

Example 7.6 (Multiplying du by a constant). Evaluate $\int x(x^2 + 4)^7 \, dx$.

Solution. We let $u = x^2 + 4$, because the composite $u^7 = (x^2 + 4)^7$ appears in the integrand. The differential $du = 2x \, dx$ does not appear as is, but we can multiply by $\frac{1}{2}$ to obtain $\frac{1}{2}du = x \, dx$. Now

$$\int (x^2 + 4)^7 x \, dx = \int u^7 \cdot \frac{1}{2} \, du = \frac{1}{16} u^8 + C = \frac{1}{16} (x^2 + 4)^8 + C.$$

Example 7.7. Evaluate $\int \frac{(x^2 + x) \, dx}{(2x^3 + 3x^2 + 5)^4}$.

Solution. The appearance of $(2x^3 + 3x^2 + 5)^{-4}$ suggests that we try $u = 2x^3 + 3x^2 + 5$. With this choice,

$$du = (6x^2 + 6x) dx = 6(x^2 + x) dx \implies \frac{1}{6} du = (x^2 + x) dx,$$

and so

$$\int \frac{(x^2 + x) dx}{(2x^3 + 3x^2 + 5)^4} = \int \underbrace{(2x^3 + 3x^2 + 5)^{-4}}_{u^{-4}} \underbrace{(x^2 + x) dx}_{\frac{1}{6} du} = \frac{1}{6} \int u^{-4} du,$$

which equals $\frac{1}{6} \cdot \frac{u^{-3}}{-3} + C = -\frac{1}{18}(2x^3 + 3x^2 + 5)^{-3} + C$.

Comment. An integration method that works for a given function may fail if the function is changed even slightly. If we replace $x^2 + x$ by $x^2 + 1.1x$ in the previous example, substitution no longer works, because $(x^2 + 1.1x) dx$ is not a multiple of $du = (6x^2 + 6x) dx$.

Example 7.8. Evaluate $\int \cos(4\theta - 3) d\theta$.

Solution. Let $u = 4\theta - 3$. Then $du = 4 d\theta$, so $\frac{1}{4} du = d\theta$, and

$$\int \underbrace{\cos(4\theta - 3)}_{\cos u} \underbrace{d\theta}_{\frac{1}{4} du} = \frac{1}{4} \int \cos u du = \frac{1}{4} \sin u + C = \frac{1}{4} \sin(4\theta - 3) + C.$$

Example 7.9. Evaluate $\int e^{-7t} dt$.

Solution. Use the substitution $u = -7t$, $du = -7 dt$. Then $dt = -\frac{1}{7} du$, so

$$\int e^{-7t} dt = \int e^u \left(-\frac{1}{7} du\right) = -\frac{1}{7} e^u + C = -\frac{1}{7} e^{-7t} + C.$$

Antiderivatives of $\sin \theta$ and $\cos \theta$ came to us from the corresponding derivative formulas, but we have not yet found an antiderivative for $\tan \theta$. We can obtain one now using substitution.

Example 7.10 (Integral of $\tan \theta$). Evaluate $\int \tan \theta d\theta$.

Solution. The idea is to write $\tan \theta d\theta = \frac{\sin \theta d\theta}{\cos \theta}$ and to note that if $u = \cos \theta$, then $du = -\sin \theta d\theta$, so $-du = \sin \theta d\theta$:

$$\int \tan \theta d\theta = \int \frac{\sin \theta d\theta}{\cos \theta} = - \int \frac{du}{u} = -\ln |u| + C = -\ln |\cos \theta| + C.$$

Alternate forms of this antiderivative are obtained using the identity $-\ln u = \ln \frac{1}{u}$: since $-\ln |\cos \theta| = \ln \frac{1}{|\cos \theta|}$, we have

$$\int \tan \theta d\theta = \ln |\sec \theta| + C.$$

A similar approach shows that $\int \cot \theta \, d\theta = \ln |\sin \theta| + C$.

Example 7.11 (An additional step is necessary). Evaluate $\int x\sqrt{3x+2} \, dx$.

Solution. Since $\sqrt{3x+2}$ appears, we are tempted to set $u = 3x+2$. Then $du = 3 \, dx$, so $\sqrt{3x+2} \, dx = \frac{1}{3}\sqrt{u} \, du$. Unfortunately, the integrand is not $\sqrt{3x+2}$ but $x\sqrt{3x+2}$. To take care of the extra factor of x , we solve $u = 3x+2$ to obtain $x = \frac{1}{3}(u-2)$. Then

$$x\sqrt{3x+2} \, dx = \frac{1}{3}(u-2) \cdot \frac{1}{3}\sqrt{u} \, du = \frac{1}{9}(u^{3/2} - 2u^{1/2}) \, du,$$

and so

$$\begin{aligned} \int x\sqrt{3x+2} \, dx &= \frac{1}{9} \int (u^{3/2} - 2u^{1/2}) \, du = \frac{1}{9} \left(\frac{2}{5}u^{5/2} - \frac{4}{3}u^{3/2} \right) + C \\ &= \frac{2}{45}(3x+2)^{5/2} - \frac{4}{27}(3x+2)^{3/2} + C. \end{aligned}$$

Remark 7.12. Substitution does not always work, even when the integral looks relatively simple. For example, $\int \sin(x^2) \, dx$ cannot be evaluated explicitly by substitution, or by any other method. With experience one learns to recognize when substitution is likely to succeed.

7.2 Change of variables formula for definite integrals

The change of variables formula applies to definite integrals provided that the limits of integration are changed as well.

Theorem 7.13 (Change of variables for definite integrals). *If u' is continuous on $[a, b]$ and f is continuous on the range of u , then*

$$\int_a^b f(u(x))u'(x) \, dx = \int_{u(a)}^{u(b)} f(u) \, du.$$

Idea. If F is an antiderivative of f , then $F(u(x))$ is an antiderivative of $f(u(x))u'(x)$ by the chain rule. FTC I shows that both integrals are equal to $F(u(b)) - F(u(a))$. $\square$

Remark 7.14. The new limits of integration are the values of u at the old ones: as x varies from a to b , the variable $u = u(x)$ varies from $u(a)$ to $u(b)$.

Example 7.15. Evaluate $\int_0^2 x(x^2+1)^3 \, dx$.

Solution. We use the substitution $u = x^2 + 1$. Thus $du = 2x \, dx$, and therefore $x \, dx = \frac{1}{2}du$. By Theorem 7.13, the new limits of integration are

$$u(0) = 0^2 + 1 = 1 \quad \text{and} \quad u(2) = 2^2 + 1 = 5.$$

Thus

$$\int_0^2 x(x^2+1)^3 \, dx = \frac{1}{2} \int_1^5 u^3 \, du = \frac{u^4}{8} \Big|_1^5 = \frac{625-1}{8} = 78.$$

This substitution shows that the area in (A) of Figure 8 is equal to one half of the area in (B).

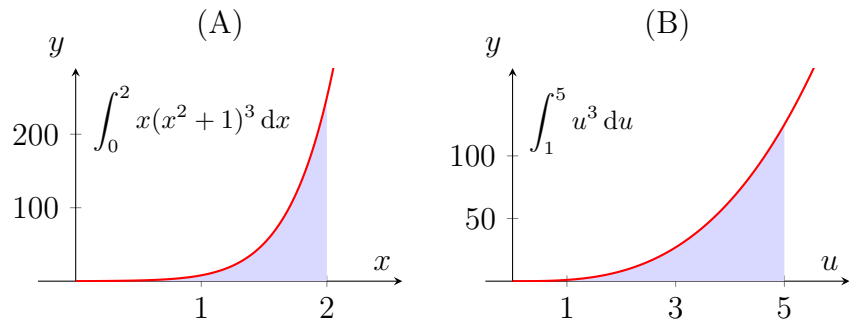


Figure 8: The substitution $u = x^2 + 1$ shows that the area in (A) is half the area in (B).

Remark 7.16. In the previous example we can avoid changing the limits of integration by evaluating the indefinite integral in terms of x :

$$\int x(x^2 + 1)^3 dx = \frac{1}{2} \int u^3 du = \frac{u^4}{8} + C = \frac{(x^2 + 1)^4}{8} + C,$$

which leads to the same result: $\int_0^2 x(x^2 + 1)^3 dx = \left. \frac{(x^2 + 1)^4}{8} \right|_0^2 = 78$.

Example 7.17. Evaluate $\int_0^{\pi/4} \tan^2 \theta \sec^2 \theta d\theta$.

Solution. The substitution $u = \tan \theta$ makes sense because $du = \sec^2 \theta d\theta$, and therefore $u^2 du = \tan^2 \theta \sec^2 \theta d\theta$. The new limits of integration are

$$u(0) = \tan 0 = 0 \quad \text{and} \quad u\left(\frac{\pi}{4}\right) = \tan \frac{\pi}{4} = 1.$$

Thus

$$\int_0^{\pi/4} \tan^2 \theta \sec^2 \theta d\theta = \int_0^1 u^2 du = \left. \frac{u^3}{3} \right|_0^1 = \frac{1}{3}.$$

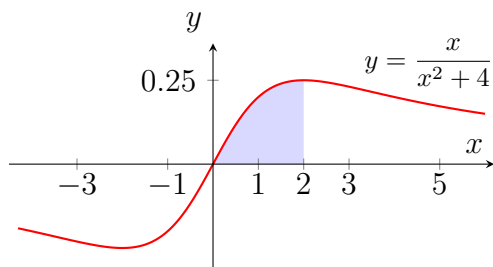


Figure 9: The area under the graph of $y = \frac{x}{x^2 + 4}$ over $[0, 2]$.

Example 7.18. Calculate the area under the graph of $y = \frac{x}{x^2 + 4}$ over $[0, 2]$.

Solution. The area (Figure 9) is equal to $\int_0^2 \frac{x \, dx}{x^2+4}$. We use the substitution

$$u = x^2 + 4, \quad du = 2x \, dx, \quad \frac{1}{2} \frac{du}{u} = \frac{x \, dx}{x^2 + 4}.$$

The new limits of integration are $u(0) = 0^2 + 4 = 4$ and $u(2) = 2^2 + 4 = 8$, so

$$\int_0^2 \frac{x \, dx}{x^2 + 4} = \frac{1}{2} \int_4^8 \frac{du}{u} = \frac{1}{2} \ln |u| \Big|_4^8 = \frac{1}{2} \ln 8 - \frac{1}{2} \ln 4 = \frac{1}{2} \ln 2 \approx 0.347.$$

Comment. There are two ways to compute a definite integral by a change of variables: change the limits of integration to the corresponding limits in u and compute the definite integral in u , or compute the antiderivative first, express it in terms of x , and then evaluate it at the original limits.

8 Area between two curves

This section covers §6.1 of the textbook.

Sometimes we are interested in the area of the region between two curves rather than the area under a single one. For instance, if two graphs record a projected quantity under two different scenarios, the area of the region between them measures the total difference between the scenarios.

Suppose we are given two functions $y = f(x)$ and $y = g(x)$ with $f(x) \geq g(x)$ for all x in an interval $[a, b]$.

Definition 8.1. A region is *vertically simple* if every vertical line that meets it does so in a single point or a single vertical line segment whose lower endpoint lies on the graph of $y = g(x)$ and whose upper endpoint lies on the graph of $y = f(x)$.

For such a region, the graph of f lies above the graph of g , and the region between them is obtained by removing the region under $y = g(x)$ from the region under $y = f(x)$ (Figure 10). Hence the area between the graphs is the area under the top function minus the area under the bottom function:

$$\text{area between the graphs} = \int_a^b f(x) \, dx - \int_a^b g(x) \, dx = \int_a^b (f(x) - g(x)) \, dx. \quad (8.1)$$

Example 8.2. Find the area of the region between the graphs of

$$f(x) = x^2 - 2x + 6, \quad g(x) = 2x - x^2, \quad 0 \leq x \leq 3.$$

Solution. Figure 11 shows that $f(x) \geq g(x)$ over $[0, 3]$, so the region is vertically simple. By (8.1), the area between the graphs is

$$\begin{aligned} \int_0^3 (f(x) - g(x)) \, dx &= \int_0^3 ((x^2 - 2x + 6) - (2x - x^2)) \, dx \\ &= \int_0^3 (2x^2 - 4x + 6) \, dx = \left(\frac{2}{3}x^3 - 2x^2 + 6x \right) \Big|_0^3 = 18. \end{aligned}$$

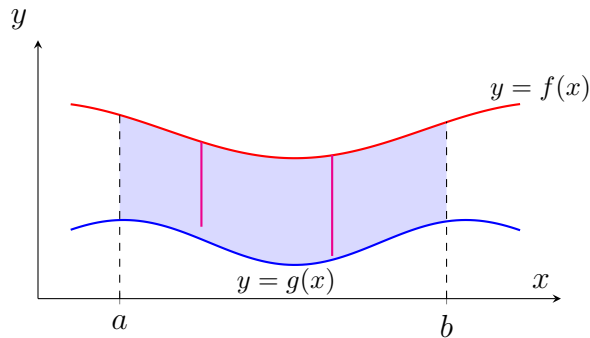


Figure 10: A vertically simple region: every vertical line meets it in a single segment running from the graph of g up to the graph of f .

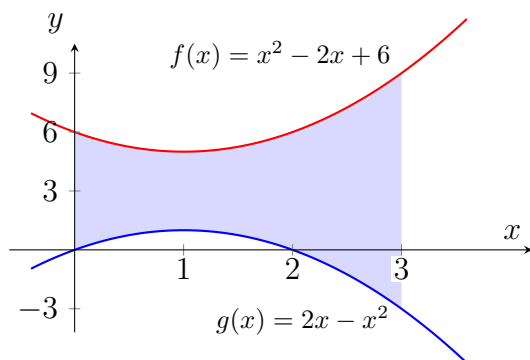


Figure 11: The region between the graphs of $f(x) = x^2 - 2x + 6$ and $g(x) = 2x - x^2$ over $[0, 3]$.

Remark 8.3. Formula (8.1) remains valid whenever $f(x) \geq g(x)$, even if f and g are not assumed to be positive. We can simply shift both functions up by adding to each a constant C large enough that both are positive over $[a, b]$; this does not change the area between them, and

$$\int_a^b ((f(x) + C) - (g(x) + C)) \, dx = \int_a^b (f(x) - g(x)) \, dx.$$

Writing $y_{\text{top}} = f(x)$ for the top curve and $y_{\text{bot}} = g(x)$ for the bottom curve, we obtain

$$\text{area between the graphs} = \int_a^b (y_{\text{top}} - y_{\text{bot}}) \, dx = \int_a^b (f(x) - g(x)) \, dx. \quad (8.2)$$

If the graphs cross, the top curve changes, and we must break the region into pieces.

Example 8.4. Find the area between the graphs of $f(x) = x^2 - 2x - 8$ and $g(x) = 2x - 8$ over $[-2, 4]$.

Solution. First we must determine which graph lies on top.

Step 1. Sketch the region and find the points of intersection. The graph of f is a parabola and the graph of g is a line (Figure 12). The graphs intersect where $f(x) - g(x) = 0$, and

$$f(x) - g(x) = (x^2 - 2x - 8) - (2x - 8) = x^2 - 4x = x(x - 4),$$

so they intersect at $x = 0$ and $x = 4$.

Step 2. Set up the integrals and evaluate. From the sketch, $f(x) \geq g(x)$ on $[-2, 0]$ and $g(x) \geq f(x)$ on $[0, 4]$. This tells us to subdivide the region into two vertically simple regions and to write the area as a sum of integrals:

$$\begin{aligned} \int_{-2}^4 (y_{\text{top}} - y_{\text{bot}}) \, dx &= \int_{-2}^0 (f(x) - g(x)) \, dx + \int_0^4 (g(x) - f(x)) \, dx \\ &= \int_{-2}^0 (x^2 - 4x) \, dx + \int_0^4 (4x - x^2) \, dx \\ &= \left(\frac{x^3}{3} - 2x^2 \right) \Big|_{-2}^0 + \left(2x^2 - \frac{x^3}{3} \right) \Big|_0^4 = \frac{32}{3} + \frac{32}{3} = \frac{64}{3}. \end{aligned}$$

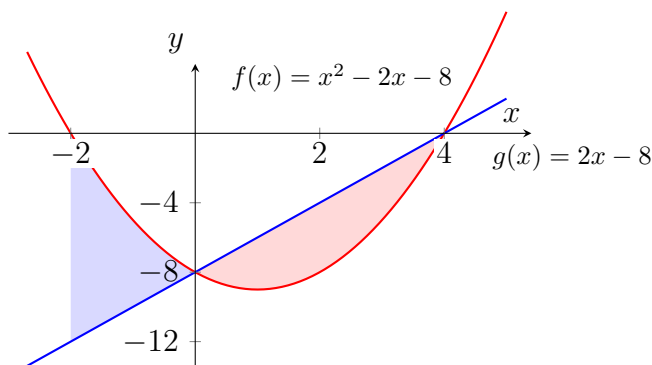


Figure 12: The graphs of $f(x) = x^2 - 2x - 8$ and $g(x) = 2x - 8$ cross at $x = 0$, so the region over $[-2, 4]$ splits into two vertically simple pieces.

Example 8.5 (Calculating area by dividing the region). Find the area of the region bounded by the graphs of $y = 27/x^2$, $y = 27x$, and $y = x$.

Solution. *Step 1. Sketch the region and find the points of intersection.* The curve $y = 27/x^2$ cuts off a region in the sector between the two lines $y = 27x$ and $y = x$ (Figure 13). We find the intersection of $y = 27/x^2$ and $y = 27x$ by solving

$$\frac{27}{x^2} = 27x \implies x^3 = 1 \implies x = 1,$$

and the intersection of $y = 27/x^2$ and $y = x$ by solving

$$\frac{27}{x^2} = x \implies x^3 = 27 \implies x = 3.$$

Step 2. Set up the integrals and evaluate. Figure 13 shows that $y_{\text{bot}} = x$ throughout, but y_{top} changes at $x = 1$ from $y_{\text{top}} = 27x$ to $y_{\text{top}} = 27/x^2$. So the region is not vertically simple, and we break it into two parts A and B , each vertically simple, computing their areas

separately:

$$\begin{aligned}\text{area of } A &= \int_0^1 (y_{\text{top}} - y_{\text{bot}}) \, dx = \int_0^1 (27x - x) \, dx = 13x^2 \Big|_0^1 = 13, \\ \text{area of } B &= \int_1^3 (y_{\text{top}} - y_{\text{bot}}) \, dx = \int_1^3 \left(\frac{27}{x^2} - x \right) \, dx = \left(-\frac{27}{x} - \frac{x^2}{2} \right) \Big|_1^3 = 14.\end{aligned}$$

The total area bounded by the curves is the sum $13 + 14 = 27$.

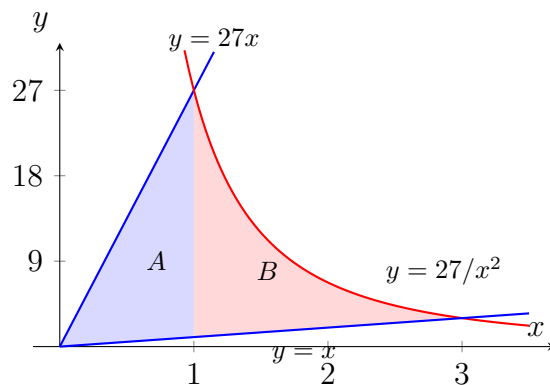


Figure 13: The region bounded by $y = 27/x^2$, $y = 27x$, and $y = x$ as a sum of two areas.

8.1 Integration along the y -axis

Suppose we are given x as a function of y , say $x = g(y)$. What is the meaning of $\int_c^d g(y) \, dy$? This integral can be interpreted as signed area, where regions to the *right* of the y -axis have positive area and regions to the *left* have negative area:

$$\int_c^d g(y) \, dy = \text{signed area between the graph and the } y\text{-axis for } c \leq y \leq d.$$

Example 8.6. For the region between the curve $x = y^2 - 4$ and the y -axis with $-4 \leq y \leq 4$ (Figure 14), the part of the shaded region to the left of the y -axis has negative signed area. The signed area of the entire region is

$$\underbrace{\int_{-4}^4 (y^2 - 4) \, dy}_{\substack{\text{area to the right of } y\text{-axis minus} \\ \text{area to the left of } y\text{-axis}}} = \left(\frac{1}{3}y^3 - 4y \right) \Big|_{-4}^4 = \frac{32}{3}.$$

More generally, suppose $g(y) \geq h(y)$, so that the graph of $x = g(y)$ lies to the right of the graph of $x = h(y)$. In this case we write $x_{\text{right}} = g(y)$ and $x_{\text{left}} = h(y)$.

Definition 8.7. A region is *horizontally simple* if every horizontal line that meets it in more than a single point does so in a single line segment whose left endpoint lies on the curve $x = h(y)$ and whose right endpoint lies on the curve $x = g(y)$.

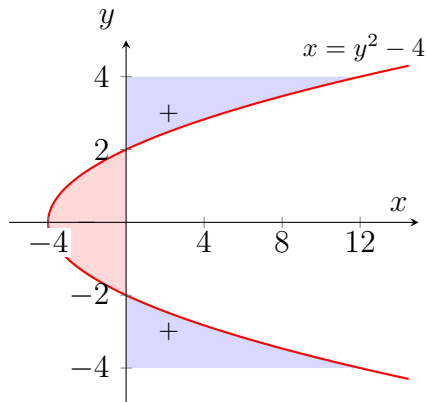


Figure 14: The region between $x = y^2 - 4$ and the y -axis. The part to the left of the y -axis contributes negative signed area.

The formula for the area of such a region, corresponding to (8.2), is

$$\text{area between the graphs} = \int_c^d (x_{\text{right}} - x_{\text{left}}) dy = \int_c^d (g(y) - h(y)) dy. \quad (8.3)$$

Example 8.8. Calculate the area enclosed by the graphs of $h(y) = y^2 - 2$ and $g(y) = y^2 - \frac{1}{4}y^4 + 2$.

Solution. Figure 15 shows that the enclosed region stretches between the two points of intersection of the graphs, and that $g(y) \geq h(y)$ over the region. Therefore the region is horizontally simple with $x_{\text{right}} = g(y)$ and $x_{\text{left}} = h(y)$. To set up the integral we determine the points of intersection by solving $g(y) = h(y)$ for y :

$$y^2 - \frac{1}{4}y^4 + 2 = y^2 - 2 \implies \frac{1}{4}y^4 = 4 \implies y = \pm 2.$$

Now

$$x_{\text{right}} - x_{\text{left}} = (y^2 - \frac{1}{4}y^4 + 2) - (y^2 - 2) = 4 - \frac{1}{4}y^4,$$

so the enclosed area is

$$\int_{-2}^2 (x_{\text{right}} - x_{\text{left}}) dy = \int_{-2}^2 \left(4 - \frac{1}{4}y^4\right) dy = \left(4y - \frac{y^5}{20}\right) \Big|_{-2}^2 = \frac{32}{5} - \left(-\frac{32}{5}\right) = \frac{64}{5}.$$

Remark 8.9. It would be more difficult to calculate the area of the region in Figure 15 as an integral with respect to x , because the curves are not graphs of functions of x .

For many regions we have a choice between integrating with respect to x and integrating with respect to y . The decision is usually based on two factors:

- how easy it is to obtain the curves as functions of one variable in terms of the other;
- how easy it is to subdivide the region into simple regions and to integrate the functions involved.

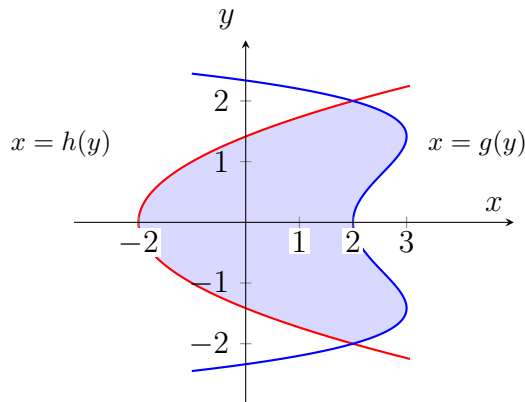


Figure 15: The horizontally simple region enclosed by $x = h(y) = y^2 - 2$ and $x = g(y) = y^2 - \frac{1}{4}y^4 + 2$.

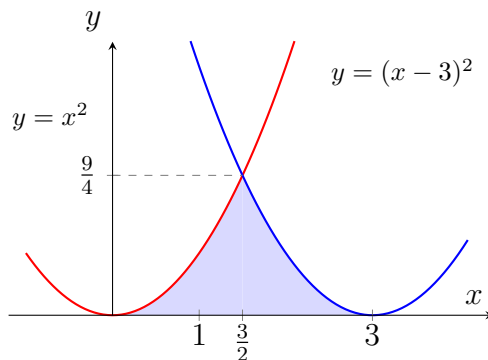


Figure 16: This region is horizontally simple, but it is easier to cut it into two vertically simple regions.

The next example carries out the computation both ways.

Example 8.10. Find the area of the region bounded by the three curves $y = x^2$, $y = (x-3)^2$, and $y = 0$.

Solution. The region appears in Figure 16. It is not vertically simple, since the top function changes over the interval $[0, 3]$: the two parabolas meet where $x^2 = (x-3)^2$, that is, at $x = \frac{3}{2}$. Splitting the region into two vertically simple pieces,

$$\int_0^{3/2} x^2 dx + \int_{3/2}^3 (x-3)^2 dx = \left. \frac{x^3}{3} \right|_0^{3/2} + \left. \frac{(x-3)^3}{3} \right|_{3/2}^3 = \frac{9}{8} + \frac{9}{8} = \frac{9}{4}.$$

Now let us redo the problem using the fact that the region is horizontally simple. To determine x_{right} and x_{left} we invert the formulas for each of the parabolas. The left boundary of the region is the right side of the parabola $y = x^2$: solving for x gives $x = \pm\sqrt{y}$, and the right side corresponds to the positive choice, so $x_{\text{left}} = \sqrt{y}$. The right boundary is the left side of the parabola $y = (x-3)^2$: solving for x gives $x = 3 \pm \sqrt{y}$, and the left side is obtained

by choosing the minus sign, so $x_{\text{right}} = 3 - \sqrt{y}$. The two curves meet at $y = \frac{9}{4}$, so the area is

$$\int_0^{9/4} ((3 - \sqrt{y}) - \sqrt{y}) \, dy = \int_0^{9/4} (3 - 2y^{1/2}) \, dy = \left(3y - \frac{4}{3}y^{3/2} \right) \Big|_0^{9/4} = \frac{27}{4} - \frac{9}{2} = \frac{9}{4},$$

as before.

9 Setting up integrals: volume, average value

This section covers §6.2 of the textbook.

Which quantities are represented by integrals? Roughly speaking, integrals represent quantities that are the “total amount” of something, such as area, volume, or mass. Each such quantity can be approximated by a sum obtained by dividing an object into small pieces over which the quantity is easy to compute, and the exact value is obtained by passing to a limit. This gives a two-step procedure that we will use repeatedly:

- (1) approximate the quantity by a sum of N terms;
- (2) pass to the limit as $N \rightarrow \infty$ to obtain an integral.

9.1 Volume

Our first example is the *volume* of a solid body. Recall that the volume of a *right cylinder* is Ah , where A is the area of the base and h is the height, measured perpendicular to the base. Here “right cylinder” is meant in the general sense: the base need not be circular, but the sides are perpendicular to the base.

Suppose that a solid body extends from height $y = a$ to height $y = b$ along the y -axis, and that the areas of its *horizontal cross sections* (the intersections of the solid with horizontal planes) vary from level to level. Let $A(y)$ be the cross-sectional area at height y .

To compute the volume V of the body, divide it into N horizontal slices of thickness $\Delta y = (b - a)/N$, where the i th slice extends from y_{i-1} to y_i with $y_i = a + i\Delta y$ (Figure 17). Let V_i be the volume of the i th slice. If N is very large, then Δy is very small and the slices are very thin; in this case the i th slice is nearly a right cylinder of base $A(y_{i-1})$ and height Δy , so $V_i \approx A(y_{i-1})\Delta y$. Summing the volumes of the slices,

$$V = \sum_{i=1}^N V_i \approx \sum_{i=1}^N A(y_{i-1})\Delta y.$$

The sum on the right is a left-endpoint approximation to $\int_a^b A(y) \, dy$. If A is continuous, the approximation improves as $N \rightarrow \infty$ and converges to this integral. We conclude that *the volume of a solid is the integral of its cross-sectional area*.

Theorem 9.1 (Volume as the integral of cross-sectional area). *Let $A(y)$ be the area of the horizontal cross section at height y of a solid body extending from $y = a$ to $y = b$. Then*

$$\text{volume of the solid body} = \int_a^b A(y) \, dy. \tag{9.1}$$

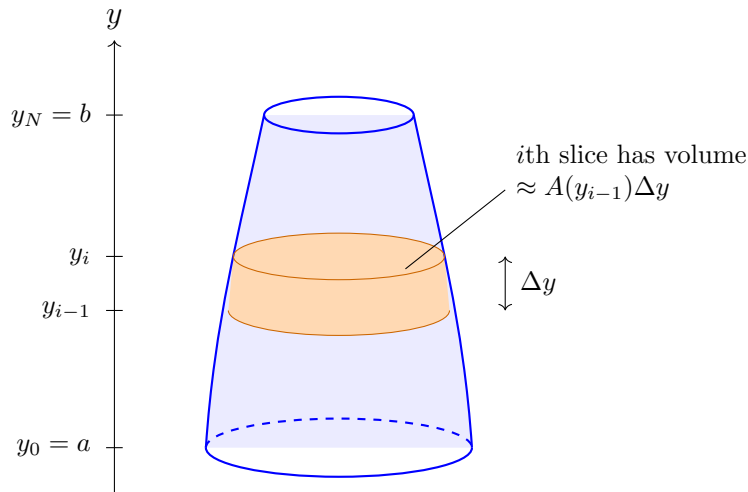


Figure 17: Dividing a solid into thin horizontal slices. Each slice is nearly a right cylinder, whose volume is its cross-sectional area times its thickness.

Example 9.2 (Volume of a pyramid). Calculate the volume V of a pyramid of height 9 m whose base is a square of side 6 m.

Solution. *Step 1. Find a formula for $A(y)$.* Figure 18 shows that the horizontal cross section at height y is a square, of side s say. To find s , apply the law of similar triangles to the triangle of height 9 whose base has length 3 (half the side of the base square) and to the triangle of height $9 - y$ whose base of length $\frac{1}{2}s$ lies on the cross section:

$$\frac{\text{base}}{\text{height}} = \frac{3}{9} = \frac{\frac{1}{2}s}{9-y} \implies s = \frac{2}{3}(9-y).$$

Therefore $A(y) = s^2 = \frac{4}{9}(9-y)^2$.

Step 2. Compute V as the integral of $A(y)$. By (9.1),

$$V = \int_0^9 A(y) \, dy = \int_0^9 \frac{4}{9}(9-y)^2 \, dy = -\frac{4}{27}(9-y)^3 \Big|_0^9 = 108 \text{ m}^3.$$

The antiderivative $-\frac{4}{27}(9-y)^3$ comes from the substitution $u = 9-y$; the relation $du = -dy$ introduces the negative sign. This agrees with the formula $V = \frac{1}{3}Ah$ for the volume of a pyramid of base A and height h , since $\frac{1}{3}(6^2)(9) = 108$.

Example 9.3. Compute the volume V of the solid whose base is the region between the parabola $y = 9 - x^2$ and the x -axis, and whose vertical cross sections perpendicular to the y -axis are semicircles.

Solution. To find a formula for the area $A(y)$ of the cross section, observe that $y = 9 - x^2$ can be written as $x = \pm\sqrt{9-y}$. We see in Figure 19 that the cross section at height y is a semicircle of radius $r = \sqrt{9-y}$, whose area is

$$A(y) = \frac{1}{2}\pi r^2 = \frac{\pi}{2}(9-y).$$

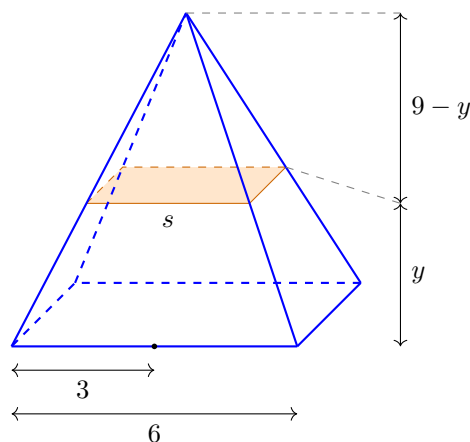


Figure 18: A horizontal cross section of the pyramid is a square of side $s = \frac{2}{3}(9 - y)$.

Therefore

$$V = \int_0^9 A(y) dy = \frac{\pi}{2} \int_0^9 (9 - y) dy = \frac{\pi}{2} \left(9y - \frac{1}{2}y^2 \right) \Big|_0^9 = \frac{81\pi}{4}.$$

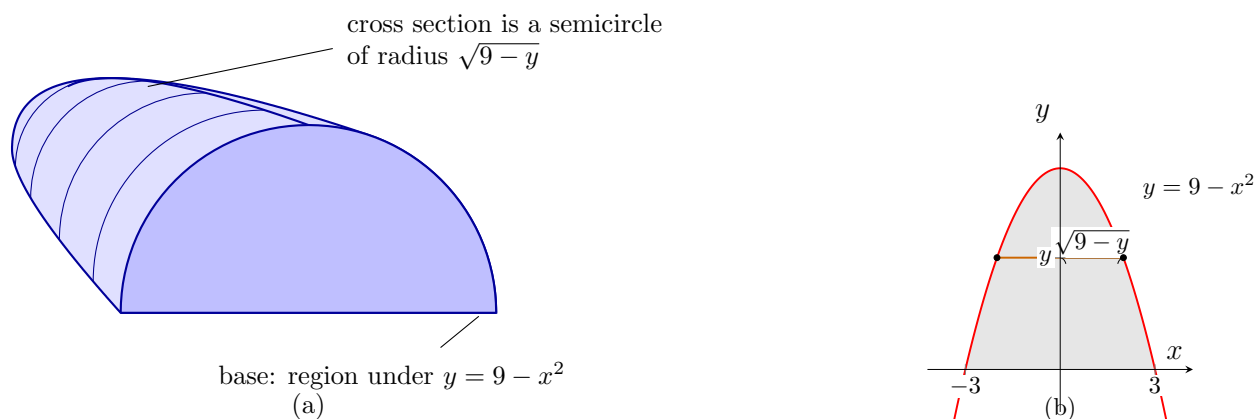


Figure 19: (a) The solid whose base is the region under $y = 9 - x^2$ and whose cross sections perpendicular to the y -axis are semicircles. (b) At height y the base is a segment of half-width $\sqrt{9 - y}$, which is the radius of the semicircular cross section there.

In the next example we compute a volume using vertical rather than horizontal cross sections, which leads to an integral with respect to x rather than y .

Example 9.4 (Volume of a sphere: vertical cross sections). Compute the volume of a sphere of radius R .

Solution. As we see in Figure 20, the vertical cross section of the sphere at x is a circle whose radius r satisfies $x^2 + r^2 = R^2$, that is, $r = \sqrt{R^2 - x^2}$. Note that R is the fixed radius of the sphere, while r varies as the cross sections vary. The area of the cross section

is $A(x) = \pi r^2 = \pi(R^2 - x^2)$, so the sphere has volume

$$\int_{-R}^R \pi(R^2 - x^2) dx = \pi \left(R^2 x - \frac{x^3}{3} \right) \Big|_{-R}^R = 2 \left(\pi R^3 - \frac{\pi R^3}{3} \right) = \frac{4}{3} \pi R^3.$$

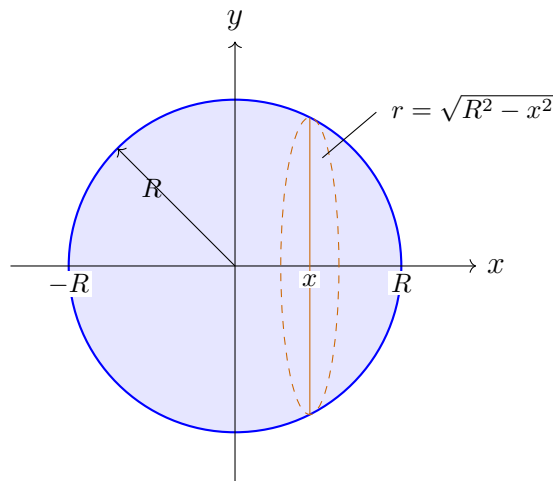


Figure 20: The vertical cross section of a sphere of radius R at x is a circle of radius $r = \sqrt{R^2 - x^2}$.

Comment. *Cavalieri's principle* states that solids with equal cross-sectional areas have equal volume. Formula (9.1) contains this principle, because the volumes are certainly equal if the cross-sectional areas $A(y)$ are equal. A stack of coins illustrates the point: sliding the coins sideways changes the shape of the stack but not its volume.

9.2 Average value

Recall that the average of N numbers $a_1, \dots, a_N$ is their sum divided by N . We cannot define the average value of a function f on an interval $[a, b]$ as a sum, since there are infinitely many values of x to consider. But the right-endpoint approximation

$$R_N = \frac{b-a}{N} (f(x_1) + f(x_2) + \dots + f(x_N))$$

divided by $b-a$ is exactly the average of the equally spaced function values $f(x_i)$:

$$\frac{1}{b-a} R_N = \underbrace{\frac{f(x_1) + f(x_2) + \dots + f(x_N)}{N}}_{\text{average of the function values}}.$$

For large N it is reasonable to think of this quantity as an approximation to the average of f on $[a, b]$, so we define the average value itself as the limit.

Definition 9.5. The *average value* of an integrable function f on $[a, b]$ is the quantity

$$\text{average value} = \lim_{N \rightarrow \infty} \frac{1}{b-a} R_N = \frac{1}{b-a} \int_a^b f(x) \, dx.$$

It is also called the *mean value* of f on $[a, b]$.

Observation 9.6. The average value M of a nonnegative function is the average height of its graph: the region under the graph has the same area as the rectangle of height M over $[a, b]$ (Figure 21), because $\int_a^b f(x) \, dx = M(b-a)$.

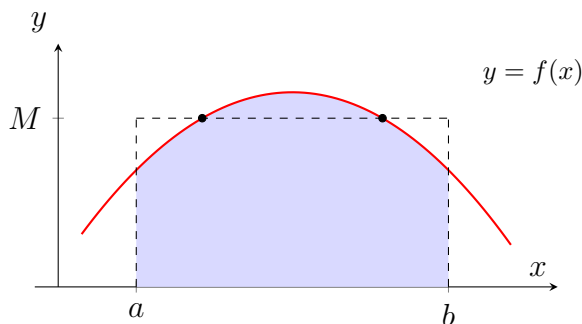


Figure 21: The area under the graph equals the area of the rectangle whose height is the average value M . The graph meets the top edge of the rectangle at the points where f takes on its average value.

Example 9.7. Find the average value of $f(x) = \sqrt{x}$ on $[0, 9]$.

Solution. The average value is

$$\frac{1}{9} \int_0^9 \sqrt{x} \, dx = \frac{1}{9} \cdot \frac{2}{3} x^{3/2} \Big|_0^9 = \frac{2}{27} \cdot 27 = 2.$$

This is reasonable, because $\sqrt{x}$ varies from 0 to 3 on $[0, 9]$ and the average lies between these two extremes.

Example 9.8 (Vertical jump of a bushbaby). The bushbaby (*Galago senegalensis*) is a small primate with remarkable jumping ability. Find the average speed during a jump whose initial vertical velocity is $v_0 = 500$ cm/s, using Galileo's formula $h(t) = v_0 t - \frac{1}{2} g t^2$ for the height in centimeters, where $g = 980$ cm/s².

Solution. The height is $h(t) = t(v_0 - \frac{1}{2} g t)$, which is zero at $t = 0$ and at $t = 2v_0/g = \frac{1000}{980} = \frac{50}{49}$ seconds, when the jump ends. The velocity is $h'(t) = v_0 - g t = 500 - 980t$, which is negative for $t > v_0/g = \frac{25}{49}$, so the speed is $|h'(t)| = |500 - 980t|$. As we see in Figure 22, the integral of the speed is the sum of the areas of two triangles of base $\frac{25}{49}$ and height 500:

$$\int_0^{50/49} |500 - 980t| \, dt = 2 \cdot \frac{1}{2} \left(\frac{25}{49} \right) (500) = \frac{12,500}{49}.$$

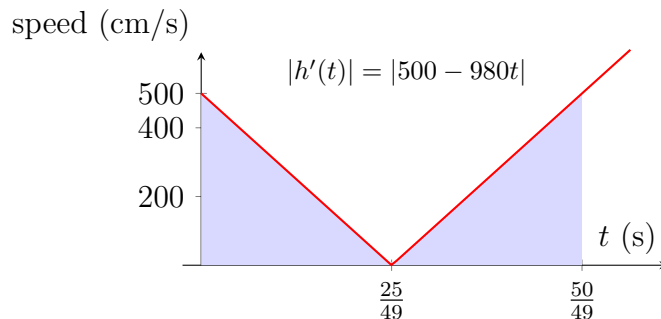


Figure 22: The graph of the speed during the jump. The distance traveled is the total area of the two triangles.

The average speed $\bar{s}$ is therefore

$$\bar{s} = \frac{1}{\frac{50}{49}} \int_0^{50/49} |500 - 980t| dt = \frac{49}{50} \cdot \frac{12,500}{49} = 250 \text{ cm/s.}$$

Remark 9.9. There is an important difference between the average of a list of numbers and the average value of a continuous function. If the average score on an exam is 84, then 84 lies between the highest and lowest scores, but it is possible that no student received a score of 84. By contrast, a continuous function always takes on its average value somewhere in the interval.

Theorem 9.10 (Mean value theorem for integrals). *If f is continuous on $[a, b]$, then there exists a value c in $[a, b]$ such that*

$$f(c) = \frac{1}{b-a} \int_a^b f(x) dx.$$

Idea. Let M be the average value. Since f is continuous, it attains a minimum value $m_{\min}$ and a maximum value $M_{\max}$ on $[a, b]$, and the bounds of Theorem 2.6 give $m_{\min}(b-a) \leq \int_a^b f(x) dx \leq M_{\max}(b-a)$. Dividing by $b-a$ shows that $m_{\min} \leq M \leq M_{\max}$, so the intermediate value theorem provides a c with $f(c) = M$. Graphically, the horizontal line $y = M$ cannot lie entirely above or entirely below the graph, so it must cross it (Figure 21). $\square$

Example 9.11. The average value of $f(x) = \sqrt{x}$ on $[0, 9]$ is 2, and indeed $f(4) = 2$ with 4 in $[0, 9]$.

10 Volumes of revolution: disks and washers

This section covers §6.3 of the textbook. We use the words *rotate* and *revolve* interchangeably.

Definition 10.1. A *solid of revolution* is a solid obtained by rotating a region of the plane about an axis. The region sweeps out the solid as it turns.

Spheres and right circular cones are familiar examples: a half-disk rotated about its straight edge sweeps out a sphere, and a right triangle rotated about one of its legs sweeps out a cone.

10.1 The disk method

Let f be continuous with $f(x) \geq 0$ on $[a, b]$, and rotate the region under the graph of f about the x -axis. The resulting solid has a special feature: all of its cross sections perpendicular to the x -axis are circles. Indeed, the cross section at x is the circle swept out by the vertical segment joining $(x, 0)$ to $(x, f(x))$, so it is a circle of radius $R = f(x)$ and its area is

$$\text{area of the cross section at } x = \pi R^2 = \pi f(x)^2.$$

The volume of a solid is the integral of its cross-sectional area (Theorem 9.1), and the same argument applies with the roles of the two axes exchanged: if $A(x)$ is the area of the cross section perpendicular to the x -axis, then $V = \int_a^b A(x) dx$. Integrating the areas of the circular cross sections gives the following formula, called the *disk method* because the slices of the solid are circular disks.

Method (Volume of revolution: disks). If f is continuous and $f(x) \geq 0$ on $[a, b]$, then the solid obtained by rotating the region under the graph of f about the x -axis has volume

$$V = \pi \int_a^b R^2 dx = \pi \int_a^b f(x)^2 dx. \quad (10.1)$$

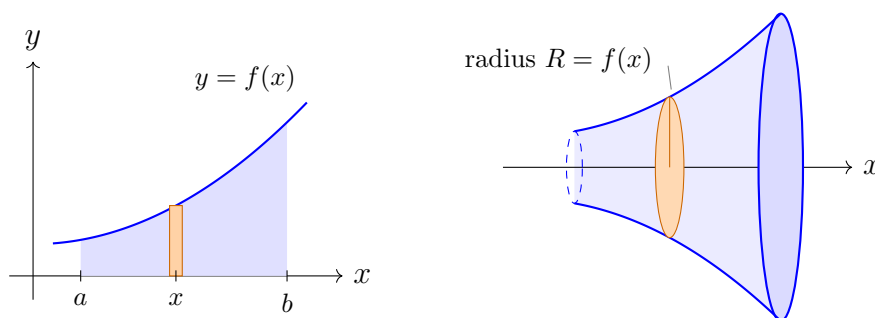


Figure 23: Rotating the region under the graph of f about the x -axis. The vertical slice at x sweeps out a disk of radius $f(x)$.

Example 10.2. Compute the volume V of the solid obtained by rotating the region under the graph of $y = x^3$ about the x -axis for $0 \leq x \leq 1$.

Solution. By (10.1) with $f(x) = x^3$,

$$V = \pi \int_0^1 (x^3)^2 dx = \pi \int_0^1 x^6 dx = \pi \cdot \frac{x^7}{7} \Big|_0^1 = \frac{\pi}{7}.$$

10.2 Washers

There are several useful variations on (10.1). Consider first the region *between* the graphs of two functions f and g with $f(x) \geq g(x) \geq 0$ on $[a, b]$. When this region is rotated about the x -axis, the vertical segment joining $(x, g(x))$ to $(x, f(x))$ no longer reaches the axis, so it

sweeps out the region between two concentric circles rather than a full disk. Such a region is called an *annulus*, or informally a *washer* (Figure 24). Its two radii are

$$R_{\text{outer}} = f(x), \quad R_{\text{inner}} = g(x),$$

and its area is the area of the larger circle minus the area of the smaller one,

$$\pi R_{\text{outer}}^2 - \pi R_{\text{inner}}^2 = \pi(f(x)^2 - g(x)^2).$$

The volume of the solid of revolution is the integral of this cross-sectional area.

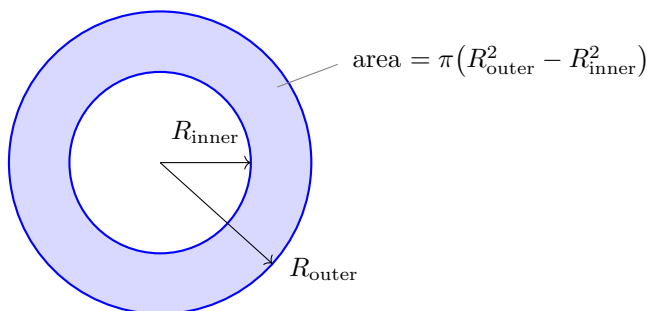


Figure 24: A washer: the region between two concentric circles. Its area is the difference of the areas of the two circles.

Method (Volume of revolution: washers). If f and g are continuous with $f(x) \geq g(x) \geq 0$ on $[a, b]$, then the solid obtained by rotating the region between the two graphs about the x -axis has volume

$$V = \pi \int_a^b (R_{\text{outer}}^2 - R_{\text{inner}}^2) dx = \pi \int_a^b (f(x)^2 - g(x)^2) dx. \quad (10.2)$$

Caution. The integrand in (10.2) is $f(x)^2 - g(x)^2$, not $(f(x) - g(x))^2$. The area of a washer is a difference of the areas of two circles, and squaring the difference of the radii gives something else entirely.

Example 10.3. Find the volume V obtained by revolving the region between $y = x^2 + 1$ and $y = 1$ about the x -axis for $0 \leq x \leq 2$.

Solution. The graph of $y = x^2 + 1$ lies above the line $y = 1$ on $[0, 2]$, so $R_{\text{outer}} = x^2 + 1$ and $R_{\text{inner}} = 1$. By (10.2),

$$\begin{aligned} V &= \pi \int_0^2 (R_{\text{outer}}^2 - R_{\text{inner}}^2) dx = \pi \int_0^2 ((x^2 + 1)^2 - 1^2) dx \\ &= \pi \int_0^2 (x^4 + 2x^2) dx = \pi \left(\frac{x^5}{5} + \frac{2}{3}x^3 \right) \Big|_0^2 = \frac{176\pi}{15}. \end{aligned}$$

10.3 Rotating about a horizontal axis

Nothing changes in principle if the axis of rotation is a horizontal line $y = c$ other than the x -axis. What changes are the radii, which are always *distances from the axis of rotation*. Since vertical distance is a difference of y -coordinates, for a region lying above the line $y = c$ we subtract c from each of the two bounding curves.

Strategy. To set up the integral for a volume of revolution, sketch the region and visualize the cross section perpendicular to the axis. It is a disk if the region touches the axis, and a washer if it does not. Then read the radii off the picture as distances to the axis.

Example 10.4. Find the volume V of the ring obtained by rotating the region between the graphs of $f(x) = x^2 + 3$ and $g(x) = 5 - x^2$ about the horizontal line $y = -1$.

Solution. First we find the points of intersection of the two graphs by solving

$$f(x) = g(x) \implies x^2 + 3 = 5 - x^2 \implies x^2 = 1 \implies x = \pm 1,$$

and Figure 25 shows that $g(x) \geq f(x)$ for $-1 \leq x \leq 1$.

Rotating about $y = -1$ instead of the x -axis lengthens both radii by 1 unit, since each radius is now measured from the line $y = -1$:

$$R_{\text{outer}} = g(x) - (-1) = 6 - x^2, \quad R_{\text{inner}} = f(x) - (-1) = x^2 + 4.$$

The volume of revolution is the integral of the area of this washer:

$$\begin{aligned} V &= \pi \int_{-1}^1 (R_{\text{outer}}^2 - R_{\text{inner}}^2) \, dx = \pi \int_{-1}^1 ((6 - x^2)^2 - (x^2 + 4)^2) \, dx \\ &= \pi \int_{-1}^1 ((36 - 12x^2 + x^4) - (x^4 + 8x^2 + 16)) \, dx \\ &= \pi \int_{-1}^1 (20 - 20x^2) \, dx = 20\pi \left(x - \frac{x^3}{3} \right) \Big|_{-1}^1 = \frac{80\pi}{3}. \end{aligned}$$

Whether the cross sections are disks or washers depends on the axis, not only on the region, as the next example shows.

Example 10.5. Find the volume obtained by rotating the region between the graph of $f(x) = 4 - x^2$ and the line $y = 4$ for $0 \leq x \leq 2$ about

- (a) the line $y = 4$;
- (b) the line $y = 6$.

Solution. In each case we ask what the cross section is: a disk or a washer? Figure 27 shows the region and the two axes.

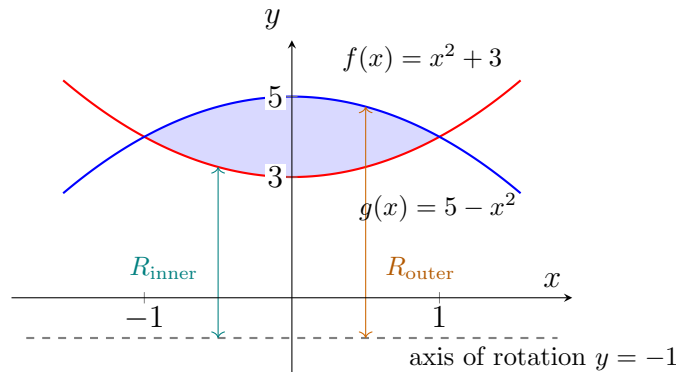


Figure 25: The region between $f(x) = x^2 + 3$ and $g(x) = 5 - x^2$, to be rotated about the line $y = -1$. Each radius is measured from the axis of rotation.

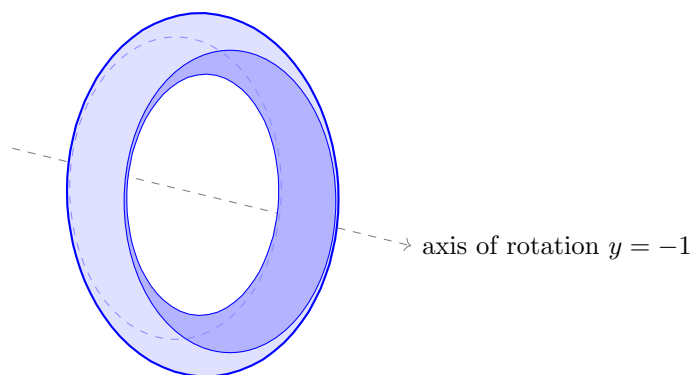


Figure 26: The ring swept out when the region of Figure 25 is rotated about the line $y = -1$. The cross sections perpendicular to the axis are the washers of radii $R_{\text{inner}} = x^2 + 4$ and $R_{\text{outer}} = 6 - x^2$; the two radii agree at $x = \pm 1$, where the ring comes to an edge.

- (a) The region touches the line $y = 4$, so the vertical segment at x sweeps out a *disk*, of radius

$$R = 4 - f(x) = 4 - (4 - x^2) = x^2.$$

Note that the length of the segment is $4 - f(x)$ rather than $f(x) - 4$, because the line $y = 4$ lies above the graph of f . The volume is

$$V = \pi \int_0^2 R^2 dx = \pi \int_0^2 x^4 dx = \pi \cdot \frac{x^5}{5} \Big|_0^2 = \frac{32\pi}{5}.$$

- (b) Rotated about the line $y = 6$, the same segment sweeps out a *washer*. Its outer radius is the distance from the graph of f to the line $y = 6$, and its inner radius is the distance from the line $y = 4$ to the line $y = 6$:

$$R_{\text{outer}} = 6 - f(x) = x^2 + 2, \quad R_{\text{inner}} = 6 - 4 = 2.$$

Therefore

$$\begin{aligned} V &= \pi \int_0^2 (R_{\text{outer}}^2 - R_{\text{inner}}^2) \, dx = \pi \int_0^2 ((x^2 + 2)^2 - 2^2) \, dx \\ &= \pi \int_0^2 (x^4 + 4x^2) \, dx = \pi \left(\frac{x^5}{5} + \frac{4}{3}x^3 \right) \Big|_0^2 = \frac{256\pi}{15}. \end{aligned}$$

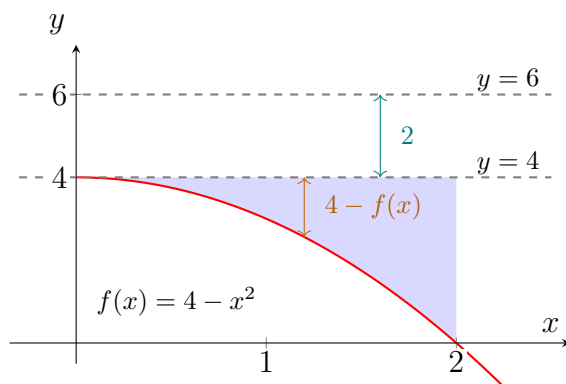


Figure 27: The region between the graph of $f(x) = 4 - x^2$ and the line $y = 4$. Rotated about $y = 4$ it generates disks, and rotated about $y = 6$ it generates washers.

10.4 Rotating about a vertical axis

The disk and washer methods work just as well for solids of revolution about a vertical axis, but the cross sections perpendicular to a vertical axis are horizontal. We therefore integrate with respect to y , and the radii must be expressed as functions of y : the bounding curves have to be described in the form $x = g(y)$.

Example 10.6. Find the volume of the solid obtained by rotating the region under the graph of $f(x) = 4 - x^2$ for $0 \leq x \leq 2$ about the vertical axis $x = -1$.

Solution. Figure 28 shows that the horizontal segment at height y sweeps out a washer when rotated about the line $x = -1$. We are going to integrate with respect to y , so we need the two radii as functions of y . Solving $y = 4 - x^2$ for x gives $x^2 = 4 - y$, that is $x = \pm\sqrt{4 - y}$; since the region lies to the right of the y -axis, we take the positive square root. The segment at height y therefore runs from $x = 0$ to $x = \sqrt{4 - y}$, and measuring distances from the line $x = -1$,

$$R_{\text{outer}} = \sqrt{4 - y} + 1, \quad R_{\text{inner}} = 1.$$

Consequently

$$R_{\text{outer}}^2 - R_{\text{inner}}^2 = (\sqrt{4 - y} + 1)^2 - 1^2 = (4 - y) + 2\sqrt{4 - y} + 1 - 1 = (4 - y) + 2\sqrt{4 - y}.$$

The region extends from $y = 0$ to $y = 4$ along the y -axis, so

$$\begin{aligned} V &= \pi \int_0^4 (R_{\text{outer}}^2 - R_{\text{inner}}^2) dy = \pi \int_0^4 ((4 - y) + 2\sqrt{4 - y}) dy \\ &= \pi \left(4y - \frac{y^2}{2} - \frac{4}{3}(4 - y)^{3/2} \right) \Big|_0^4 = \pi \left(8 + \frac{32}{3} \right) = \frac{56\pi}{3}. \end{aligned}$$

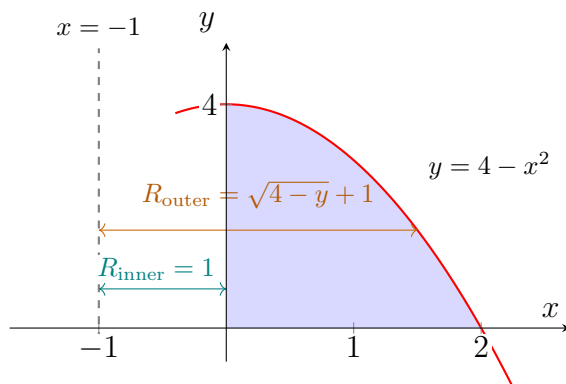


Figure 28: The horizontal segment at height y sweeps out a washer when the region is rotated about the vertical line $x = -1$.

Comment. The several volume formulas of this section are not new results. Each of them comes from the principle of §6.2 that the volume of a solid is the integral of its cross-sectional area, (9.1); all that changes from one situation to another is the shape of the cross section and the variable of integration. Even the distinction between disks and washers is only apparent, since a disk is a washer with $R_{\text{inner}} = 0$.

11 Integration by parts

This section covers §7.1 of the textbook. Substitution (Section 7) reverses the chain rule; the technique of this section reverses the product rule, and it often converts an integral we cannot evaluate into one we can.

Let u and v be differentiable functions of x . The product rule says

$$\frac{d}{dx}(uv) = u \frac{dv}{dx} + \frac{du}{dx}v,$$

so uv is an antiderivative of the right-hand side:

$$uv = \int u \frac{dv}{dx} dx + \int v \frac{du}{dx} dx.$$

Moving the second integral to the other side, and writing the differentials $du = \frac{du}{dx} dx$ and $dv = \frac{dv}{dx} dx$ as in (7.1), we obtain the following formula.

Method (Integration by parts). If u and v are differentiable functions of x , then

$$\int u \, dv = uv - \int v \, du. \quad (11.1)$$

Since the left-hand side of (11.1) is the integral of a product, integration by parts is worth considering whenever the integrand is a product of two functions. It is not, however, a product rule for antidifferentiation: it does not always produce an antiderivative of a product. It works for $x \cos x$, as in Example 11.2, but it does not work for $x \tan x$ (see the exercises of §7.1 of the textbook). In the other direction, an integrand that is not visibly a product can sometimes be handled by a clever choice of u and dv ; the standard instance is $\ln x$, in Example 11.5.

Notation 11.1. A choice of u and dv , together with the resulting du and v , is conveniently recorded in a two-by-two array:

$$\begin{array}{cc} u = \square & dv = \square \\ du = \square & v = \square \end{array}$$

The top row multiplies to the integrand we started with, the diagonal entries u and v give the product uv in (11.1), and the bottom row multiplies to the integrand $v \, du$ of the new integral.

Example 11.2. Evaluate $\int x e^{3x} \, dx$.

Solution. The integrand is a product, so we try writing $x e^{3x} \, dx = u \, dv$ with

$$\begin{array}{ll} u = x & dv = e^{3x} \, dx \\ du = dx & v = \frac{1}{3} e^{3x} \end{array}$$

where u is differentiated and dv is antidifferentiated. Then (11.1) gives

$$\int \underbrace{x}_u \underbrace{e^{3x} \, dx}_{dv} = \underbrace{\frac{1}{3} x e^{3x}}_{uv} - \int \underbrace{\frac{1}{3} e^{3x}}_v \underbrace{dx}_{du} = \frac{1}{3} x e^{3x} - \frac{1}{9} e^{3x} + C.$$

We check by differentiating:

$$\frac{d}{dx} \left(\frac{1}{3} x e^{3x} - \frac{1}{9} e^{3x} \right) = \frac{1}{3} e^{3x} + x e^{3x} - \frac{1}{3} e^{3x} = x e^{3x}.$$

Remark 11.3. In applying (11.1), any antiderivative v of dv may be used.

The key step is deciding how to write the integral in the form $\int u \, dv$. The formula expresses $\int u \, dv$ in terms of uv and $\int v \, du$, so it helps only when $\int v \, du$ is easier than the integral we started with. This suggests two guidelines:

- choose dv so that $v = \int dv$ can be evaluated;
- choose u so that $\frac{du}{dx}$ is simpler than u itself.

The choices in Example 11.2 satisfy both, and they transformed the integral into a simpler one. A bad choice would have been $u = e^{3x}$ and $dv = x \, dx$, for then $du = 3e^{3x} \, dx$ and $v = \frac{1}{2}x^2$, and (11.1) would give

$$\int x e^{3x} \, dx = \underbrace{\frac{1}{2}x^2}_{uv} e^{3x} - \int \underbrace{\left(\frac{1}{2}x^2\right)}_v \underbrace{3e^{3x} \, dx}_{du};$$

the new integral is essentially $\int x^2 e^{3x} \, dx$, which is worse than the one we had.

Example 11.4 (Integrating by parts more than once). Evaluate $\int x^2 \sin x \, dx$.

Solution. Apply integration by parts a first time with $u = x^2$ and $dv = \sin x \, dx$, so that $du = 2x \, dx$ and $v = -\cos x$:

$$\int x^2 \sin x \, dx = -x^2 \cos x + 2 \int x \cos x \, dx. \quad (11.2)$$

The remaining integral is again a product, and a second application of (11.1), this time with $u = x$ and $dv = \cos x \, dx$, gives

$$\int x \cos x \, dx = x \sin x - \int \sin x \, dx = x \sin x + \cos x + C.$$

Substituting this into (11.2),

$$\int x^2 \sin x \, dx = -x^2 \cos x + 2x \sin x + 2 \cos x + C.$$

Comment. Taking $u = x^2$ in the previous example is the sensible choice because integration by parts then replaces the integration of $x^2 \sin x$ by the integration of $2x \cos x$, in which the power of x has dropped.

The function $f(x) = \ln x$ is one of the basic functions for which we have no antiderivative formula so far. Integration by parts supplies one.

Example 11.5 (Taking $dv = dx$). Evaluate $\int \ln x \, dx$.

Solution. The integrand is not a product, so at first sight integration by parts does not apply. But we may regard $\ln x \, dx$ as the product of $\ln x$ and dx , and take

$$\begin{aligned} u &= \ln x & dv &= dx \\ du &= \frac{1}{x} dx & v &= x \end{aligned}$$

Then

$$\int \ln x \, dx = x \ln x - \int x \cdot \frac{1}{x} \, dx = x \ln x - \int dx = x \ln x - x + C.$$

Differentiating confirms this: $\frac{d}{dx}(x \ln x - x) = x \cdot \frac{1}{x} + \ln x - 1 = \ln x$.

We may therefore add to our table of integral formulas

$$\int \ln x \, dx = x \ln x - x + C.$$

Remark 11.6. The choice $dv = dx$ is surprisingly effective. Besides $\ln x$, it also works for the inverse trigonometric functions.

There is a convenient version of (11.1) for definite integrals.

Method (Integration by parts for definite integrals).

$$\int_a^b u \, dv = uv|_a^b - \int_a^b v \, du. \quad (11.3)$$

Example 11.7. Compute $\int_0^2 x e^{-x} \, dx$.

Solution. Take $u = x$ and $dv = e^{-x} \, dx$, so $du = dx$ and $v = -e^{-x}$. By (11.3),

$$\begin{aligned} \int_0^2 x e^{-x} \, dx &= (-x e^{-x})|_0^2 + \int_0^2 e^{-x} \, dx = -2e^{-2} + (-e^{-x})|_0^2 \\ &= -2e^{-2} - e^{-2} + 1 = 1 - 3e^{-2} \approx 0.594. \end{aligned}$$

Example 11.8 (Going in a circle?). Evaluate $\int e^x \sin x \, dx$.

Solution. There are two reasonable ways of writing $e^x \sin x \, dx$ as $u \, dv$. Let us try $u = \sin x$ and $dv = e^x \, dx$, so that $du = \cos x \, dx$ and $v = e^x$:

$$\int e^x \sin x \, dx = e^x \sin x - \int e^x \cos x \, dx. \quad (11.4)$$

Integrating by parts again, now with $u = \cos x$ and $dv = e^x \, dx$, so that $du = -\sin x \, dx$ and $v = e^x$,

$$\int e^x \cos x \, dx = e^x \cos x + \int e^x \sin x \, dx. \quad (11.5)$$

Equation (11.5) returns us to the integral we started with, so it looks as though we are going in a circle. But substituting (11.5) into (11.4) and solving is enough:

$$\int e^x \sin x \, dx = e^x \sin x - e^x \cos x - \int e^x \sin x \, dx.$$

Adding $\int e^x \sin x \, dx$ to both sides, and adding a constant C on the right because no integral remains there to generate one,

$$2 \int e^x \sin x \, dx = e^x(\sin x - \cos x) + C \implies \int e^x \sin x \, dx = \frac{1}{2}e^x(\sin x - \cos x) + C.$$

Comment. Dividing an arbitrary constant by 2 leaves an arbitrary constant, so we keep calling it C and absorb the $\frac{1}{2}$ into it. The choice $u = e^x$, $dv = \sin x \, dx$ works equally well in this example.

Definition 11.9. A *reduction formula* expresses an integral depending on a parameter n in terms of a similar integral with a smaller value of n . A reduction formula may have to be applied repeatedly in order to reduce an integral to one that can be computed.

Example 11.10 (A reduction formula). Derive the reduction formula

$$\int \sin^n x \, dx = -\frac{1}{n} \sin^{n-1} x \cos x + \frac{n-1}{n} \int \sin^{n-2} x \, dx, \quad (11.6)$$

and use it to evaluate $\int \sin^3 x \, dx$.

Solution. We do not know how to integrate $\sin^n x$, but we do know how to integrate $\sin x$, so we split off one factor: take $u = \sin^{n-1} x$ and $dv = \sin x \, dx$, so that $du = (n-1) \sin^{n-2} x \cos x \, dx$ and $v = -\cos x$. Then

$$\int \sin^n x \, dx = -\sin^{n-1} x \cos x + (n-1) \int \sin^{n-2} x \cos^2 x \, dx.$$

Now $\cos^2 x = 1 - \sin^2 x$, so the integral on the right splits into two, one of which is the integral we started with:

$$\int \sin^n x \, dx = -\sin^{n-1} x \cos x + (n-1) \int \sin^{n-2} x \, dx - (n-1) \int \sin^n x \, dx.$$

Adding $(n-1) \int \sin^n x \, dx$ to both sides and dividing by n gives (11.6).

Taking $n = 3$ in (11.6),

$$\int \sin^3 x \, dx = -\frac{1}{3} \sin^2 x \cos x + \frac{2}{3} \int \sin x \, dx = -\frac{1}{3} \sin^2 x \cos x - \frac{2}{3} \cos x + C.$$

Reduction formulas for $\int \cos^n x \, dx$, $\int x^n e^x \, dx$, and $\int (\ln x)^n \, dx$ are developed in the exercises of §7.1 of the textbook.

12 Polar coordinates

This section covers §11.3 of the textbook. Rectangular coordinates are not always the most convenient way to label points of the plane. When distance from a fixed point or the direction of a point matters more than its horizontal and vertical displacements, polar coordinates are more natural.

Definition 12.1. Let P be a point of the plane and let O be the origin. The *polar coordinates* of P are the pair (r, θ) , where r is the distance from P to O and θ is the angle between the segment $\overline{OP}$ and the positive x -axis. We call r the *radial coordinate* and θ the *angular coordinate* of P . By convention, an angle is positive if the corresponding rotation is counterclockwise.

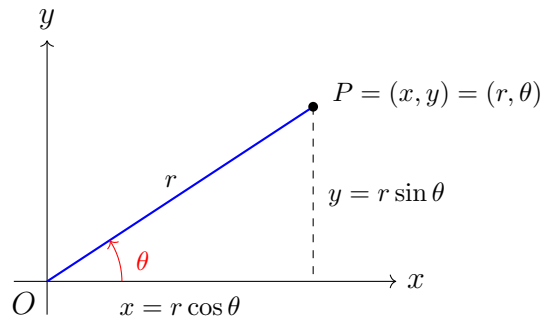


Figure 29: The polar coordinates (r, θ) of a point P , and their relation to the rectangular coordinates (x, y) .

Figure 29 exhibits the relation between the two coordinate systems. The right triangle with hypotenuse $\overline{OP}$ gives $x = r \cos \theta$ and $y = r \sin \theta$, while the distance formula gives $r^2 = x^2 + y^2$, and dividing y by x gives $\tan \theta = y/x$ when $x \neq 0$. This yields the conversion formulas

$$\begin{aligned} \text{polar to rectangular: } & x = r \cos \theta, & y &= r \sin \theta, \\ \text{rectangular to polar: } & r = \sqrt{x^2 + y^2}, & \tan \theta &= \frac{y}{x} \quad (x \neq 0). \end{aligned} \quad (12.1)$$

Caution. We do *not* write $\theta = \tan^{-1}(y/x)$, since that relation holds only for $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$.

The two families of *grid lines* of the polar coordinate system are

circles centered at $O \longleftrightarrow r = \text{constant}$, rays starting at $O \longleftrightarrow \theta = \text{constant}$,

as in Figure 30. Every point of the plane other than the origin lies on exactly one circle and one ray of these families, and the two grid lines through it determine its polar coordinates.

Example 12.2 (From polar to rectangular coordinates). Find the rectangular coordinates of the point P with polar coordinates $(6, \frac{3\pi}{4})$.

Solution. By (12.1),

$$x = r \cos \theta = 6 \cos \frac{3\pi}{4} = 6 \left(-\frac{\sqrt{2}}{2} \right) = -3\sqrt{2}, \quad y = r \sin \theta = 6 \sin \frac{3\pi}{4} = 6 \left(\frac{\sqrt{2}}{2} \right) = 3\sqrt{2}.$$

Passing from rectangular to polar coordinates requires more care, because the angular coordinate must be chosen so that (r, θ) lies in the correct quadrant. If $r > 0$, an angular coordinate of $P = (x, y)$ is

$$\theta = \begin{cases} \tan^{-1} \frac{y}{x} & \text{if } x > 0, \\ \tan^{-1} \frac{y}{x} + \pi & \text{if } x < 0, \\ \pm \frac{\pi}{2} & \text{if } x = 0. \end{cases} \quad (12.2)$$

Example 12.3 (From rectangular to polar coordinates). Find polar coordinates for the point $P = (2, 5)$.

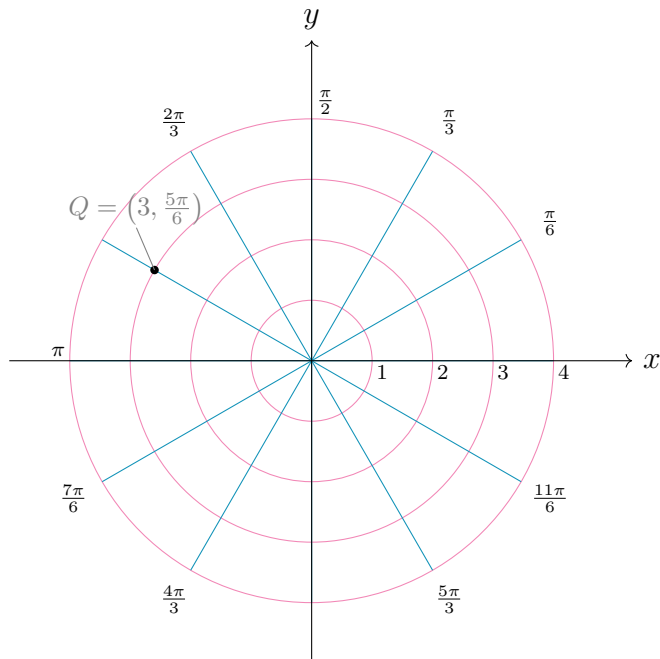


Figure 30: Grid lines in polar coordinates: the circles $r = \text{constant}$ and the rays $\theta = \text{constant}$. The point Q lies on the circle $r = 3$ and the ray $\theta = \frac{5\pi}{6}$.

Solution. Since $P = (x, y) = (2, 5)$,

$$r = \sqrt{x^2 + y^2} = \sqrt{2^2 + 5^2} = \sqrt{29} \approx 5.39, \quad \tan \theta = \frac{y}{x} = \frac{5}{2}.$$

The point P lies in the first quadrant, so $x > 0$ and (12.2) gives $\theta = \tan^{-1} \frac{5}{2} \approx 1.19$. Thus P has polar coordinates $(r, \theta) \approx (5.39, 1.19)$.

Some remarks are in order before we go on.

- The angular coordinate is not unique, because (r, θ) and $(r, \theta + 2\pi n)$ label the same point for every integer n . For instance, the point $(0, 2)$ in rectangular coordinates has radial coordinate 2, and its angular coordinate may be taken to be any of $\frac{\pi}{2}, \frac{5\pi}{2}, \dots$ or $-\frac{3\pi}{2}, -\frac{7\pi}{2}, \dots$
- The origin has no well-defined angular coordinate, so we assign to it the polar coordinates $(0, \theta)$ for every angle θ .
- By convention we allow *negative* radial coordinates: by definition, $(-r, \theta)$ is the reflection of (r, θ) through the origin (Figure 31). With this convention, $(-r, \theta)$ and $(r, \theta + \pi)$ represent the same point.
- Polar coordinates can be made unique for points other than the origin by restricting r and θ . The usual choice is $r > 0$ and $0 \leq \theta < 2\pi$, but other choices are sometimes convenient.

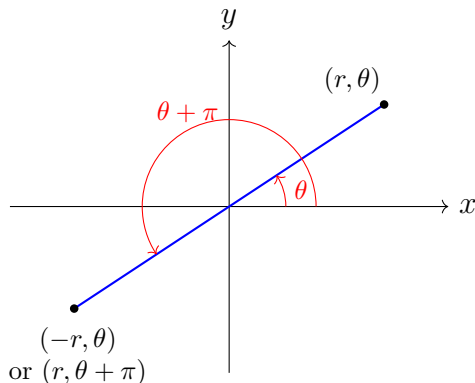


Figure 31: The point $(-r, \theta)$ is the reflection of (r, θ) through the origin, so it coincides with $(r, \theta + \pi)$.

Useful tip. There are two angles between 0 and 2π satisfying $\tan \theta = y/x$. One of them pairs with $r = \sqrt{x^2 + y^2}$ to give polar coordinates of P ; the other pairs with $-r$.

Example 12.4 (Choosing θ correctly). Find two polar representations of $P = (-2, -2)$, one with $r > 0$ and one with $r < 0$.

Solution. The two coordinates satisfy

$$r = \pm \sqrt{(-2)^2 + (-2)^2} = \pm 2\sqrt{2}, \quad \tan \theta = \frac{y}{x} = 1.$$

Now $\tan^{-1} 1 = \frac{\pi}{4}$, which points into the first quadrant, whereas P lies in the third quadrant. Since $x < 0$, (12.2) gives the correct angle for the positive radial coordinate,

$$\theta = \tan^{-1} \frac{y}{x} + \pi = \frac{\pi}{4} + \pi = \frac{5\pi}{4}.$$

If instead we use the negative radial coordinate $r = -2\sqrt{2}$, then the angle becomes $\frac{\pi}{4}$. Thus

$$P = \left(2\sqrt{2}, \frac{5\pi}{4} \right) \quad \text{or} \quad \left(-2\sqrt{2}, \frac{\pi}{4} \right).$$

12.1 Polar equations

A curve may be described in polar coordinates by an equation relating r and θ , which we call a *polar equation*. By convention, solutions with $r < 0$ are allowed.

The simplest polar equations are the ones defining the grid lines. The circle of radius R centered at the origin is $r = R$, and a line through the origin is $\theta = \theta_0$, where θ_0 is the angle between the line and the x -axis: the points with $\theta = \theta_0$ are exactly the points (r, θ_0) with r arbitrary, positive, negative, or zero, and these sweep out the whole line.

Example 12.5 (Line through the origin). Find a polar equation of the line through the origin of slope 2.

Solution. A line of slope m makes an angle θ_0 with the x -axis, where $m = \tan \theta_0$. Here $\theta_0 = \tan^{-1} 2 \approx 1.11$, so the line has polar equation $\theta = \tan^{-1} 2$, or $\theta \approx 1.11$.

A line $\mathcal{L}$ that does not pass through the origin has a unique point P_0 closest to the origin, and its polar equation can be written down in terms of the polar coordinates (d, α) of P_0 .

Proposition 12.6. *The line $\mathcal{L}$ whose point closest to the origin is $P_0 = (d, \alpha)$ has polar equation*

$$r = d \sec(\theta - \alpha). \quad (12.3)$$

Idea. The segment $\overline{OP_0}$ is perpendicular to $\mathcal{L}$, so if $P = (r, \theta)$ is any point of $\mathcal{L}$ other than P_0 , the triangle OPP_0 has a right angle at P_0 (Figure 32). Reading the cosine of the angle at O off that triangle gives $d/r = \cos(\theta - \alpha)$, which is (12.3). $\square$

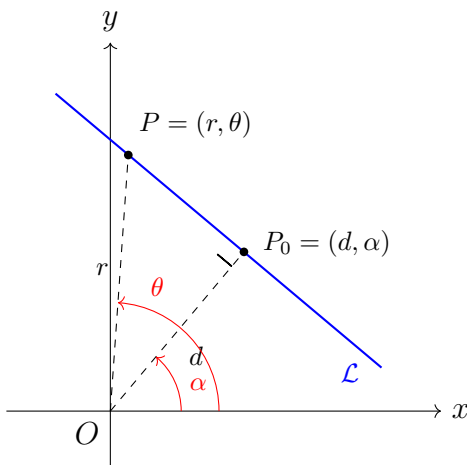


Figure 32: The point P_0 of $\mathcal{L}$ closest to the origin, and a second point P of $\mathcal{L}$. The triangle OPP_0 has a right angle at P_0 .

Example 12.7. Find the polar equation of the line $\mathcal{L}$ tangent to the circle $r = 3$ at the point with polar coordinates $P_0 = (3, \frac{\pi}{6})$.

Solution. A tangent line to a circle is perpendicular to the radius at the point of tangency, so the point of $\mathcal{L}$ closest to the origin is P_0 itself. Taking $(d, \alpha) = (3, \frac{\pi}{6})$ in (12.3) gives

$$r = 3 \sec\left(\theta - \frac{\pi}{6}\right).$$

12.2 Sketching polar curves

To sketch the graph of a polar equation $r = f(\theta)$, we can tabulate r for a convenient list of angles, plot the resulting points, and join them by a smooth curve.

Example 12.8. Sketch the curve $r = 1 + \cos \theta$.

Solution. As θ runs from 0 to 2π the function $1 + \cos \theta$ takes all of its values, and after that it repeats, so these angles suffice. Tabulating,

θ	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$	$\frac{3\pi}{4}$	π	$\frac{5\pi}{4}$	$\frac{3\pi}{2}$	$\frac{7\pi}{4}$
$r = 1 + \cos \theta$	2	1.707	1	0.293	0	0.293	1	1.707

Plotting these points and joining them gives the curve in Figure 33. It is called a *cardioid*, after the heart it resembles.

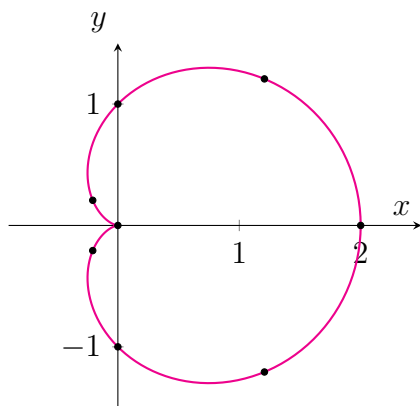


Figure 33: The cardioid $r = 1 + \cos \theta$, with the eight tabulated points marked.

Guessing the shape of a polar curve from its equation is often hard. Two further techniques help: rewriting the equation in rectangular coordinates, and exploiting symmetry.

Example 12.9 (Converting to rectangular coordinates). Identify the curve with polar equation $r = 2a \cos \theta$, where $a > 0$ is a constant.

Solution. Multiplying both sides by r makes the conversion easy, since $r^2 = x^2 + y^2$ and $r \cos \theta = x$:

$$r^2 = 2ar \cos \theta \implies x^2 + y^2 = 2ax \implies x^2 - 2ax + a^2 + y^2 = a^2,$$

that is, $(x - a)^2 + y^2 = a^2$. This is the circle of radius a centered at $(a, 0)$, which passes through the origin and is tangent there to the y -axis.

Remark 12.10. The same computation shows that $r = 2a \sin \theta$ is the circle $x^2 + (y - a)^2 = a^2$, of radius a and centered at $(0, a)$ on the y -axis.

Observation 12.11. The points (r, θ) and $(r, -\theta)$ are reflections of each other in the x -axis. Consequently, if $r = f(\theta)$ satisfies $f(-\theta) = f(\theta)$, its graph is symmetric about the x -axis, and it is enough to sketch the part with $\theta \geq 0$.

Example 12.12 (Symmetry about the x -axis). Sketch the *limaçon* $r = 2 \cos \theta - 1$.

Solution. The function $f(\theta) = 2 \cos \theta - 1$ has period 2π , so the angles $-\pi \leq \theta \leq \pi$ give the whole curve; and since $f(-\theta) = f(\theta)$, by Observation 12.11 it is enough to sketch the part with $0 \leq \theta \leq \pi$ and reflect it in the x -axis.

Rather than plot points, we read off the behavior of r from the graph of r against θ in rectangular coordinates, shown in Figure 34(A):

- as θ increases from 0 to $\frac{\pi}{3}$, the radius r decreases from 1 to 0, so the curve starts on the positive x -axis and moves in to the origin;
- as θ increases from $\frac{\pi}{3}$ to π , the radius is *negative* and decreases from 0 to -3 . A negative radius reflects the point through the origin, so the curve moves outward from the origin into the third and fourth quadrants rather than the first and second, reaching $(-3, \pi)$, the point $(3, 0)$ in rectangular coordinates, at $\theta = \pi$.

This gives the part of the curve drawn in Figure 34(B), and reflecting in the x -axis completes the limaçon.

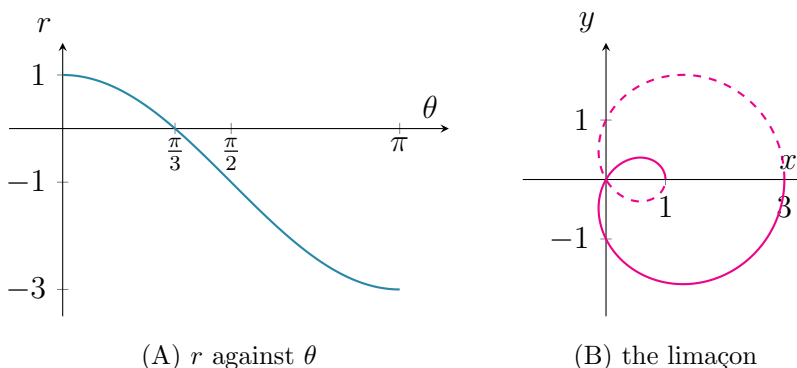


Figure 34: The limaçon $r = 2 \cos \theta - 1$. Panel (A) graphs r against θ for $0 \leq \theta \leq \pi$; panel (B) shows the corresponding part of the curve (solid) together with its reflection in the x -axis (dashed), which completes the graph.

13 Area in polar coordinates

This section covers §11.4 of the textbook. Let f be a function with $f(\theta) > 0$ for $\alpha < \theta < \beta$. The region bounded by the curve $r = f(\theta)$ and the two rays $\theta = \alpha$ and $\theta = \beta$ is a sector, as in Figure 35(A), and its area is computed by an integral in θ .

To find the formula, divide the sector into N narrow sectors of angle $\Delta\theta = (\beta - \alpha)/N$, corresponding to a partition

$$\theta_0 = \alpha < \theta_1 < \theta_2 < \cdots < \theta_N = \beta$$

of the interval $[\alpha, \beta]$, as in Figure 35(B). A circular sector of radius r and angle $\Delta\theta$ occupies the fraction $\Delta\theta/(2\pi)$ of a disk of radius r , so its area is $\frac{\Delta\theta}{2\pi} \cdot \pi r^2 = \frac{1}{2} r^2 \Delta\theta$. If $\Delta\theta$ is small, the j th narrow sector is nearly a circular sector of radius $r_j = f(\theta_j)$, so its area is approximately $\frac{1}{2} r_j^2 \Delta\theta$, and

$$\text{area of the region} \approx \sum_{j=1}^N \frac{1}{2} r_j^2 \Delta\theta = \frac{1}{2} \sum_{j=1}^N f(\theta_j)^2 \Delta\theta. \quad (13.1)$$

The right-hand side of (13.1) is a Riemann sum for $\frac{1}{2} \int_{\alpha}^{\beta} f(\theta)^2 d\theta$. If f is continuous these sums converge to the integral as $N \rightarrow \infty$, which gives the following formula.

Theorem 13.1 (Area in polar coordinates). *Let f be continuous and let $\alpha < \beta$. The area bounded by the curve $r = f(\theta)$ and the rays $\theta = \alpha$ and $\theta = \beta$ is*

$$\frac{1}{2} \int_{\alpha}^{\beta} r^2 d\theta = \frac{1}{2} \int_{\alpha}^{\beta} f(\theta)^2 d\theta. \quad (13.2)$$

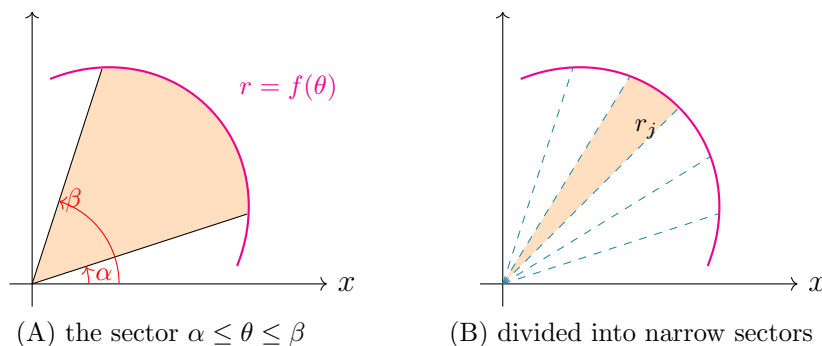


Figure 35: The area of the sector in (A) is approximated in (B) by the areas of N narrow sectors, each nearly circular of radius $r_j = f(\theta_j)$ and angle $\Delta\theta$.

As a check, $r = R$ describes the circle of radius R , and (13.2) gives

$$\frac{1}{2} \int_0^{2\pi} R^2 d\theta = \frac{1}{2} R^2 (2\pi) = \pi R^2,$$

as expected.

Caution. The integral (13.2) does *not* compute the area under a curve, as $\int_a^b f(x) dx$ does. It computes the area swept out by the radial segment from the origin as θ increases from α to β .

Applying (13.2) means integrating $f(\theta)^2$, and for the curves we meet this usually produces $\sin^2 \theta$ or $\cos^2 \theta$, neither of which we can yet antidifferentiate. The double-angle identity for the cosine supplies what is needed.

Proposition 13.2. *For all θ ,*

$$\sin^2 \theta = \frac{1}{2}(1 - \cos 2\theta), \quad \cos^2 \theta = \frac{1}{2}(1 + \cos 2\theta). \quad (13.3)$$

Idea. The double-angle identity $\cos 2\theta = \cos^2 \theta - \sin^2 \theta$, together with $\cos^2 \theta + \sin^2 \theta = 1$, gives $\cos 2\theta = 1 - 2\sin^2 \theta$ and $\cos 2\theta = 2\cos^2 \theta - 1$. Solving these for $\sin^2 \theta$ and $\cos^2 \theta$ gives (13.3). $\square$

The identities (13.3) replace a square of a trigonometric function by a constant plus a cosine, both of which we can integrate:

$$\int \sin^2 \theta d\theta = \frac{\theta}{2} - \frac{\sin 2\theta}{4} + C, \quad \int \cos^2 \theta d\theta = \frac{\theta}{2} + \frac{\sin 2\theta}{4} + C.$$

We will use (13.3) repeatedly, and we return to identities of this kind in §7.2 of the textbook.

Example 13.3. Compute the area of the right half of the disk enclosed by the circle $r = 6 \sin \theta$.

Solution. The equation $r = 6 \sin \theta$ describes the circle of radius 3 centered at $(0, 3)$, which is tangent to the x -axis at the origin. Its right half is swept out as θ varies from 0 to $\frac{\pi}{2}$, so by (13.2) and (13.3) its area is

$$\begin{aligned} \frac{1}{2} \int_0^{\pi/2} r^2 d\theta &= \frac{1}{2} \int_0^{\pi/2} (6 \sin \theta)^2 d\theta = 18 \int_0^{\pi/2} \sin^2 \theta d\theta \\ &= 18 \int_0^{\pi/2} \frac{1}{2} (1 - \cos 2\theta) d\theta = 9 \left(\theta - \frac{\sin 2\theta}{2} \right) \Big|_0^{\pi/2} = \frac{9\pi}{2}. \end{aligned}$$

This is half the area $\pi \cdot 3^2$ of the disk, as it should be.

Example 13.4. Sketch the curve $r = \cos 3\theta$ and compute the area of one petal.

Solution. We first graph r against θ in rectangular coordinates, in Figure 36(A). The radius decreases from 1 to 0 as θ goes from 0 to $\frac{\pi}{6}$ and, by symmetry, increases from 0 to 1 as θ goes from $-\frac{\pi}{6}$ to 0. So the angles $-\frac{\pi}{6} \leq \theta \leq \frac{\pi}{6}$ trace the petal labeled A in Figure 36(B). The petal B is traced for $\frac{\pi}{6} \leq \theta \leq \frac{\pi}{2}$, where $r \leq 0$, and the petal C for $\frac{\pi}{2} \leq \theta \leq \frac{5\pi}{6}$; the curve is a rose with three petals.

The area of petal A is

$$\frac{1}{2} \int_{-\pi/6}^{\pi/6} (\cos 3\theta)^2 d\theta = \frac{1}{2} \int_{-\pi/6}^{\pi/6} \frac{1 + \cos 6\theta}{2} d\theta = \left(\frac{\theta}{4} + \frac{\sin 6\theta}{24} \right) \Big|_{-\pi/6}^{\pi/6} = \frac{\pi}{12}.$$

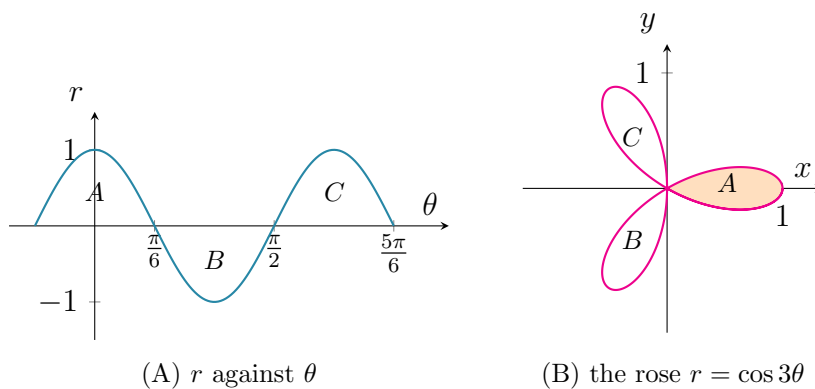


Figure 36: The rose $r = \cos 3\theta$. Each of the three petals corresponds to one arch of the graph in (A); petal A , shaded, is traced for $-\frac{\pi}{6} \leq \theta \leq \frac{\pi}{6}$.

13.1 Area between two curves

When a region is bounded by two polar curves, its area is a difference of two areas of the kind computed by (13.2), exactly as in Section 8.

Example 13.5. Find the area of the region inside the circle $r = 2 \sin \theta$ and outside the circle $r = 1$.

Solution. The two circles meet where $2 \sin \theta = 1$, that is $\sin \theta = \frac{1}{2}$, so at $\theta = \frac{\pi}{6}$ and $\theta = \frac{5\pi}{6}$ (Figure 37). Between these two angles the region in question is what remains of the sector bounded by $r = 2 \sin \theta$ after the sector bounded by $r = 1$ is removed. Therefore its area is

$$\begin{aligned} \frac{1}{2} \int_{\pi/6}^{5\pi/6} (2 \sin \theta)^2 d\theta - \frac{1}{2} \int_{\pi/6}^{5\pi/6} 1^2 d\theta &= \frac{1}{2} \int_{\pi/6}^{5\pi/6} (4 \sin^2 \theta - 1) d\theta \\ &= \frac{1}{2} \int_{\pi/6}^{5\pi/6} (2(1 - \cos 2\theta) - 1) d\theta \\ &= \frac{1}{2} (\theta - \sin 2\theta) \Big|_{\pi/6}^{5\pi/6} = \frac{\pi}{3} + \frac{\sqrt{3}}{2} \approx 1.91, \end{aligned}$$

where we used (13.3) in the second line.

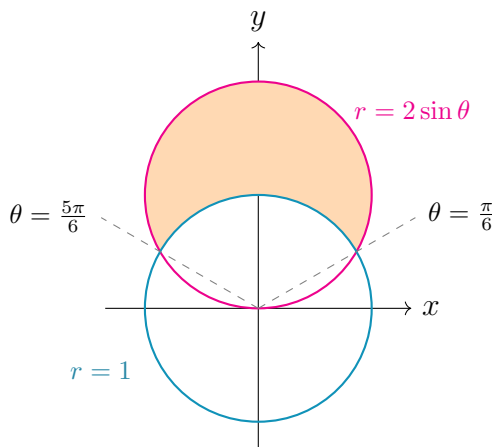


Figure 37: The shaded region lies inside the circle $r = 2 \sin \theta$ and outside the circle $r = 1$. The circles meet at the angles $\theta = \frac{\pi}{6}$ and $\theta = \frac{5\pi}{6}$.

14 Complex numbers

This section covers §1 of the supplement. Historically, each enlargement of the number system was made in order to solve equations that the previous one could not: the integers make $x + 5 = 2$ solvable, the rational numbers make $7x + 3 = 1$ solvable, and the real numbers contain the limits of sequences of rational numbers. The real number system is powerful enough to support all of calculus, but it still does not solve the equation

$$x^2 + 1 = 0.$$

To solve it one introduces a new number i with the property that $i^2 = -1$. The number i is called *imaginary*, a name that records how the idea was first received, and the number system it generates is rich enough that an entire branch of mathematics, complex analysis, is devoted to it.

14.1 Definition and basic properties

Definition 14.1. A *complex number* is a number of the form $a + bi$, equivalently $a + ib$, where a and b are real numbers and $i^2 = -1$. The real number a is called the *real part* of $a + bi$, and the real number b is called the *imaginary part* of $a + bi$. We write $\mathbb{C}$ for the set of all complex numbers.

Note that the real and imaginary parts of a complex number are themselves real numbers: the real part of $3 - 4i$ is 3 and its imaginary part is -4 , not $-4i$.

The arithmetic of complex numbers is the arithmetic of polynomials in i , subject to the one extra rule $i^2 = -1$. If $\alpha = a + bi$ and $\beta = c + di$, then

$$\alpha + \beta = (a + c) + (b + d)i, \quad \alpha - \beta = (a - c) + (b - d)i, \quad \alpha\beta = (ac - bd) + (ad + bc)i,$$

the last because $(a + bi)(c + di) = ac + adi + bci + bdi^2$ and $i^2 = -1$.

Division needs a little more thought. The quotient $\frac{\alpha}{\beta}$ can be written as $\frac{a+bi}{c+di}$, but that is not of the form $A + Bi$ with A and B real, so it is not yet visibly a complex number. To put it in that form, multiply numerator and denominator by $c - di$:

$$\frac{a + bi}{c + di} = \left(\frac{a + bi}{c + di} \right) \left(\frac{c - di}{c - di} \right) = \left(\frac{ac + bd}{c^2 + d^2} \right) + \left(\frac{-ad + bc}{c^2 + d^2} \right) i. \quad (14.1)$$

What makes this work is that $(c + di)(c - di) = c^2 + d^2$, a real number, and a positive one unless $c = d = 0$. That observation is worth a name.

Definition 14.2. Let $\alpha = a + bi$ be a complex number.

- (a) The *complex conjugate* of α is the complex number $\bar{\alpha} = a - bi$.
- (b) The *magnitude* of α , written $|\alpha|$, is $|\alpha| = \sqrt{\alpha\bar{\alpha}} = \sqrt{a^2 + b^2}$. It is also called the *modulus*, the *length*, or the *absolute value* of α .

Theorem 14.3. Let z and w be complex variables. Then

$$\overline{z + w} = \bar{z} + \bar{w}, \quad \overline{z - w} = \bar{z} - \bar{w}, \quad \overline{zw} = \bar{z}\bar{w}, \quad \overline{\left(\frac{z}{w} \right)} = \frac{\bar{z}}{\bar{w}}.$$

Idea. Conjugation may be carried out before or after the arithmetic without affecting the result. For the first identity, write $z = x + iy$ and $w = u + iv$; then $z + w = (x + u) + i(y + v)$, so the claim $\overline{z + w} = \bar{z} + \bar{w}$ says exactly that $(x + u) - i(y + v) = (x - iy) + (u - iv)$. The other three are verified in the same way. $\square$

Notation 14.4. There are conventions about which letters to use. Where a and b usually denote real numbers, α and β usually denote complex numbers; and where x and y denote real variables, z and w denote complex variables.

14.2 Geometric properties

Real numbers are represented geometrically by the points of a line, the real number line. Complex numbers are represented by the points of a plane, which we call the *complex plane*: the number $a + ib$ corresponds to the point with coordinates (a, b) , as on the left of Figure 38. Complex numbers thus make the set of points of the plane into a number system.

Three geometric facts are worth recording.

- The conjugate $\bar{\alpha} = a - bi$ is the reflection of $\alpha = a + bi$ in the x -axis (Figure 38, right). In particular $|\bar{\alpha}| = |\alpha|$.
- The magnitude $|\alpha|$ is the distance from the origin to the point (a, b) .
- Addition is vector addition: $\alpha + \beta$ is the fourth vertex of the parallelogram determined by 0, α , and β (Figure 39). Subtraction has a similar interpretation.

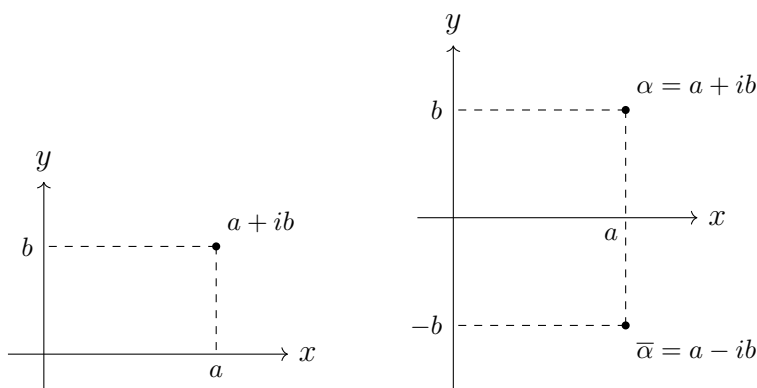


Figure 38: On the left, the complex number $a + ib$ as the point (a, b) of the complex plane. On the right, $\bar{\alpha}$ as the reflection of α in the x -axis.

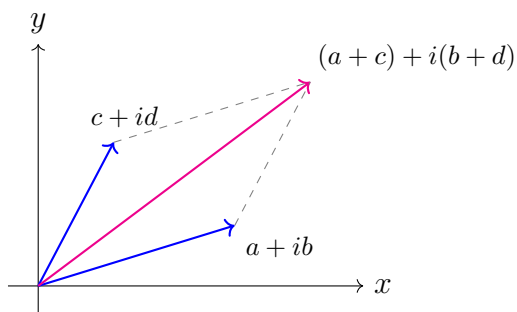


Figure 39: Addition of complex numbers is vector addition: the sum is the fourth vertex of a parallelogram.

Multiplication also has a geometric meaning, but describing it requires polar coordinates.

14.3 Polar form

A complex number $z = x + iy$ is a point (x, y) of the plane, and every point of the plane can be written in polar coordinates (r, θ) with $r \geq 0$, where $x = r \cos \theta$ and $y = r \sin \theta$ as in Section 12. Hence every complex number can be written as

$$z = r(\cos \theta + i \sin \theta), \quad (14.2)$$

which is called the *polar form* of z . Here $r = |z|$, since

$$|z| = \sqrt{(r \cos \theta)^2 + (r \sin \theta)^2} = r,$$

and the angle θ is called the *argument* of z , written $\theta = \arg(z)$.

Caution. The argument is not uniquely determined: if θ is an argument of z , then so is $\theta + 2k\pi$ for every integer k . Strictly speaking $\arg(z)$ is a whole class of angles, but no confusion arises as long as one remembers that θ and $\theta + 2k\pi$ are the same angle.

Example 14.5. Write $\sqrt{3} + i$ in polar form.

Solution. First, $r = |\sqrt{3} + i| = \sqrt{(\sqrt{3})^2 + 1^2} = 2$. Then $\cos \theta = \frac{\sqrt{3}}{2}$ and $\sin \theta = \frac{1}{2}$, so we may take $\arg(z) = \frac{\pi}{6}$. The polar form is therefore

$$\sqrt{3} + i = 2 \left(\cos \frac{\pi}{6} + i \sin \frac{\pi}{6} \right).$$

Now let $z_1 = r_1(\cos \theta_1 + i \sin \theta_1)$ and $z_2 = r_2(\cos \theta_2 + i \sin \theta_2)$. Multiplying them out,

$$z_1 z_2 = r_1 r_2 \{ [\cos \theta_1 \cos \theta_2 - \sin \theta_1 \sin \theta_2] + i [\sin \theta_1 \cos \theta_2 + \cos \theta_1 \sin \theta_2] \},$$

and the addition formulas for the sine and cosine turn the two brackets into $\cos(\theta_1 + \theta_2)$ and $\sin(\theta_1 + \theta_2)$:

$$z_1 z_2 = r_1 r_2 (\cos(\theta_1 + \theta_2) + i \sin(\theta_1 + \theta_2)). \quad (14.3)$$

A similar computation gives

$$\frac{z_1}{z_2} = \frac{r_1}{r_2} (\cos(\theta_1 - \theta_2) + i \sin(\theta_1 - \theta_2)). \quad (14.4)$$

Observation 14.6. Under multiplication, magnitudes multiply and arguments add: multiplying by $z_2 = r_2(\cos \theta_2 + i \sin \theta_2)$ scales by the factor r_2 and rotates by the angle θ_2 (Figure 40). Under division, magnitudes divide and arguments subtract, which is no surprise, division being the inverse operation of multiplication.

Example 14.7. Let $\alpha = 1 + i$. By what angle does multiplication by $\beta = -\frac{\sqrt{3}}{2} + \frac{1}{2}i$ rotate α ? Compute $\alpha\beta$ directly, without using polar form, and check that the result has the same length as α and is rotated by $\arg(\beta)$.

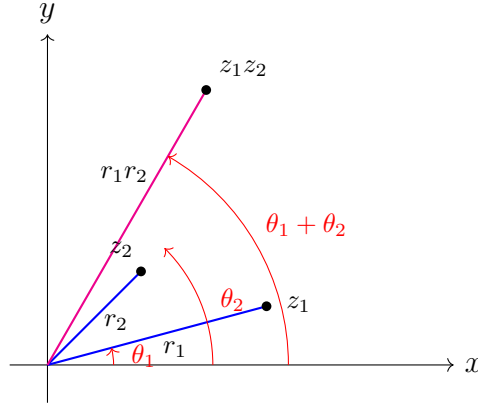


Figure 40: Multiplying z_1 by z_2 scales the magnitude by r_2 and rotates by the angle θ_2 .

Solution. Since $\beta = \cos \frac{5\pi}{6} + i \sin \frac{5\pi}{6}$, multiplication by β rotates by $\frac{5\pi}{6}$ and, $|\beta|$ being 1, does not change lengths. Multiplying out,

$$\alpha\beta = (1+i) \left(-\frac{\sqrt{3}}{2} + \frac{1}{2}i \right) = -\frac{\sqrt{3}+1}{2} - \frac{\sqrt{3}-1}{2}i,$$

so

$$|\alpha\beta| = \frac{1}{2} \sqrt{(\sqrt{3}+1)^2 + (\sqrt{3}-1)^2} = \sqrt{2} = \sqrt{1^2 + 1^2} = |\alpha|,$$

as predicted. As for the rotation, $\alpha = \sqrt{2}(\cos \frac{\pi}{4} + i \sin \frac{\pi}{4})$ and $\frac{\pi}{4} + \frac{5\pi}{6} = \frac{13\pi}{12}$, so we must check that $\alpha\beta = \sqrt{2}(\cos \frac{13\pi}{12} + i \sin \frac{13\pi}{12})$. The half-angle identities give

$$\cos \frac{13\pi}{12} = -\frac{\sqrt{2+\sqrt{3}}}{2}, \quad \sin \frac{13\pi}{12} = -\frac{\sqrt{2-\sqrt{3}}}{2},$$

both negative because $\frac{13\pi}{12}$ lies in the third quadrant, and therefore

$$\sqrt{2} \left(\cos \frac{13\pi}{12} + i \sin \frac{13\pi}{12} \right) = -\frac{\sqrt{4+2\sqrt{3}}}{2} - \frac{\sqrt{4-2\sqrt{3}}}{2}i.$$

This agrees with the direct computation, since $(\sqrt{3}+1)^2 = 4+2\sqrt{3}$ and $(\sqrt{3}-1)^2 = 4-2\sqrt{3}$.

14.4 de Moivre's theorem

If magnitudes multiply and arguments add, then raising a complex number to an integer power n raises its magnitude to the power n and multiplies its argument by n . That is de Moivre's theorem.

Theorem 14.8 (de Moivre). *Let $z = r(\cos \theta + i \sin \theta)$ and let n be an integer. Then*

$$z^n = r^n (\cos(n\theta) + i \sin(n\theta)). \quad (14.5)$$

Idea. For positive n this is repeated application of (14.3):

$$z^2 = z \cdot z = r^2(\cos 2\theta + i \sin 2\theta), \quad z^3 = z^2 \cdot z = r^3(\cos 3\theta + i \sin 3\theta),$$

and so on, each step adding one more θ to the argument. For $n = 0$ both sides equal 1. For negative exponents, (14.3) gives

$$z^{-n} z^n = r^{-n+n}(\cos(-n\theta + n\theta) + i \sin(-n\theta + n\theta)) = 1,$$

so $z^{-n} = r^{-n}(\cos(-n\theta) + i \sin(-n\theta))$ is the reciprocal of z^n , as (14.5) requires. $\square$

Example 14.9. Compute $(1 + \sqrt{3}i)^8$ and write the result in the form $a + bi$.

Solution. Here $|1 + \sqrt{3}i| = \sqrt{1^2 + (\sqrt{3})^2} = 2$, so

$$1 + \sqrt{3}i = 2 \left(\frac{1}{2} + \frac{\sqrt{3}}{2}i \right) = 2 \left(\cos \frac{\pi}{3} + i \sin \frac{\pi}{3} \right),$$

and (14.5) gives

$$(1 + \sqrt{3}i)^8 = 2^8 \left(\cos \frac{8\pi}{3} + i \sin \frac{8\pi}{3} \right) = 256 \left(-\frac{1}{2} + \frac{\sqrt{3}}{2}i \right) = -128 + 128\sqrt{3}i.$$

De Moivre's theorem also produces the roots of a complex number. An n th root of z is a complex number w with $w^n = z$. Writing both in polar form, $z = r(\cos \theta + i \sin \theta)$ and $w = s(\cos \varphi + i \sin \varphi)$, the equation $w^n = z$ becomes

$$s^n(\cos(n\varphi) + i \sin(n\varphi)) = r(\cos \theta + i \sin \theta),$$

so $s^n = r$ and $n\varphi = \theta + 2\pi k$ for some integer k . Since r and s are nonnegative and it is enough to take φ in $[0, 2\pi)$, this proves the following.

Theorem 14.10. Let $z = r(\cos \theta + i \sin \theta)$ and let n be a positive integer. Then z has the n distinct n th roots

$$w_k = r^{1/n} \left(\cos \left(\frac{\theta + 2\pi k}{n} \right) + i \sin \left(\frac{\theta + 2\pi k}{n} \right) \right), \quad k = 0, 1, \dots, n-1,$$

where $r^{1/n}$ is the positive n th root of r .

Observation 14.11. Every root has the same magnitude, $|w_k| = r^{1/n}$, and the arguments of consecutive roots differ by

$$\arg(w_{k+1}) - \arg(w_k) = \frac{2\pi}{n}.$$

The n th roots of z therefore lie equally spaced around the circle of radius $r^{1/n}$.

Example 14.12. Find the sixth roots of -64 .

Solution. In polar form $-64 = 2^6(\cos \pi + i \sin \pi)$, so by Theorem 14.10 the sixth roots are

$$w_k = 2 \left(\cos \left(\frac{\pi + 2\pi k}{6} \right) + i \sin \left(\frac{\pi + 2\pi k}{6} \right) \right), \quad k = 0, 1, 2, 3, 4, 5,$$

that is,

$$\begin{aligned} w_0 &= 2 \left(\cos \frac{\pi}{6} + i \sin \frac{\pi}{6} \right) = \sqrt{3} + i, & w_3 &= 2 \left(\cos \frac{7\pi}{6} + i \sin \frac{7\pi}{6} \right) = -\sqrt{3} - i, \\ w_1 &= 2 \left(\cos \frac{\pi}{2} + i \sin \frac{\pi}{2} \right) = 2i, & w_4 &= 2 \left(\cos \frac{3\pi}{2} + i \sin \frac{3\pi}{2} \right) = -2i, \\ w_2 &= 2 \left(\cos \frac{5\pi}{6} + i \sin \frac{5\pi}{6} \right) = -\sqrt{3} + i, & w_5 &= 2 \left(\cos \frac{11\pi}{6} + i \sin \frac{11\pi}{6} \right) = \sqrt{3} - i. \end{aligned}$$

They lie equally spaced on the circle of radius 2 (Figure 41).

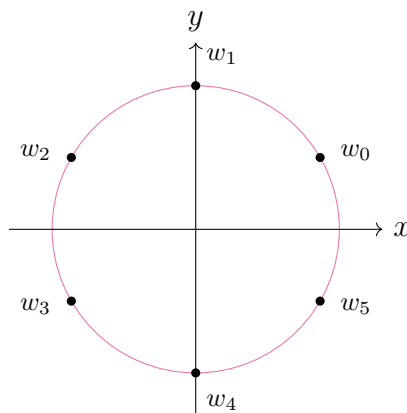


Figure 41: The sixth roots of -64 lie equally spaced on the circle of radius 2.

15 Complex exponentials

This section covers §2 of the supplement. How should e^{a+bi} be defined, where a and b are real? Whatever the definition, we want the properties of the exponential that make it useful to survive. The two most basic are that

$$e^{\alpha+\beta} = e^{\alpha}e^{\beta} \quad \text{and} \quad \frac{d}{dx}e^{\alpha x} = \alpha e^{\alpha x} \quad (15.1)$$

for all complex numbers α and β . The following definition produces a function with both properties.

Definition 15.1. The *complex exponential* is defined by

$$e^{a+bi} = e^a (\cos b + i \sin b) = e^a \cos b + i e^a \sin b \quad (15.2)$$

for real numbers a and b .

Taking $a = 0$ in (15.2) gives *Euler's formula*: for every real number x ,

$$e^{ix} = \cos x + i \sin x. \quad (15.3)$$

Theorem 15.2. *If α and β are complex numbers, then $e^{\alpha+\beta} = e^\alpha e^\beta$.*

Proof. Write $\alpha = a + ib$ and $\beta = h + ik$. Then

$$\begin{aligned} e^\alpha e^\beta &= e^a(\cos b + i \sin b) e^h(\cos k + i \sin k) \\ &= e^a e^h ([\cos b \cos k - \sin b \sin k] + i[\cos b \sin k + \sin b \cos k]) \\ &= e^{a+h} (\cos(b+k) + i \sin(b+k)) = e^{[a+h+i(b+k)]} = e^{\alpha+\beta}, \end{aligned}$$

where the third equality uses the addition formulas for the sine and cosine. $\square$

Remark 15.3. Comparing (15.3) with the polar form (14.2), every complex number can be written as $z = re^{i\theta}$ with $r = |z|$ and $\theta = \arg(z)$. In this notation the multiplication rule (14.3) is just $r_1 e^{i\theta_1} \cdot r_2 e^{i\theta_2} = r_1 r_2 e^{i(\theta_1+\theta_2)}$, an instance of Theorem 15.2, and de Moivre's theorem (14.5) is the statement that $(re^{i\theta})^n = r^n e^{in\theta}$.

Formulas for the trigonometric functions in terms of the exponential come out of Euler's formula easily. Replacing x by $-x$ in (15.3) gives $e^{-ix} = \cos x - i \sin x$, so we have two equations in $\cos x$ and $\sin x$,

$$\cos x + i \sin x = e^{ix}, \quad \cos x - i \sin x = e^{-ix}.$$

Adding them and dividing by 2 isolates $\cos x$; subtracting them and dividing by $2i$ isolates $\sin x$.

Theorem 15.4 (Exponential form of sine and cosine). *For every x ,*

$$\cos x = \frac{e^{ix} + e^{-ix}}{2}, \quad \sin x = \frac{e^{ix} - e^{-ix}}{2i}. \quad (15.4)$$

Setting $x = z = a+bi$ in (15.4) gives formulas for the sine and cosine of a complex number, and there is a good deal we can do with them. Two consequences are worth mentioning now.

- Every other trigonometric function is a rational function of the sine and the cosine, so (15.4) gives exponential formulas for all of them.
- The hyperbolic and trigonometric functions are related: $\cosh(ix) = \cos x$ and $\sinh(ix) = i \sin x$.

15.1 Complex exponentials yield trigonometric identities

Together with $e^{z+w} = e^z e^w$, the formulas (15.4) imply the three identities

$$\begin{aligned} \sin^2 \alpha + \cos^2 \alpha &= 1, \\ \sin(\alpha + \beta) &= \sin \alpha \cos \beta + \cos \alpha \sin \beta, \\ \cos(\alpha + \beta) &= \cos \alpha \cos \beta - \sin \alpha \sin \beta, \end{aligned}$$

and these three are the basis from which the other trigonometric identities are derived. So identities can be obtained from the exponential formulas and Theorem 15.2 alone, with no appeal to geometry. Here are two examples.

Example 15.5. Show that $\cos^2 x + \sin^2 x = 1$.

Solution. By (15.4),

$$\begin{aligned}\left(\frac{e^{ix} + e^{-ix}}{2}\right)^2 + \left(\frac{e^{ix} - e^{-ix}}{2i}\right)^2 &= \frac{1}{4} \left[(e^{ix})^2 + 2 + (e^{-ix})^2 + \frac{(e^{ix})^2 - 2 + (e^{-ix})^2}{i^2} \right] \\ &= \frac{1}{4} [2 + 2] = 1,\end{aligned}$$

where we used $i^2 = -1$ to change the sign of the second squared term.

Example 15.6. Show that $\sin 2x = 2 \sin x \cos x$.

Solution. Again by (15.4),

$$\sin 2x = \frac{e^{i2x} - e^{-i2x}}{2i} = \frac{1}{2i} \left[(e^{ix})^2 - (e^{-ix})^2 \right] = 2 \cdot \frac{e^{ix} - e^{-ix}}{2i} \cdot \frac{e^{ix} + e^{-ix}}{2} = 2 \sin x \cos x,$$

the middle step being the factorization of a difference of two squares.

16 Trigonometric integrals

This section covers §7.2 of the textbook. We investigate integrals of products of powers of trigonometric functions, which can usually be computed by combining substitution and integration by parts with trigonometric identities. We begin with integrals of the form

$$\int \sin^m x \cos^n x \, dx,$$

where m and n are whole numbers. The easier case is the one in which at least one of m and n is *odd*.

Example 16.1 (Odd power of $\sin x$). Evaluate $\int \sin^3 x \, dx$.

Solution. We computed this integral in Section 11 from the reduction formula (11.6), but the following method is more broadly applicable. Since $\sin^3 x$ is an odd power, split off one factor of $\sin x$ and use $\sin^2 x = 1 - \cos^2 x$ to write the rest of the integrand in terms of $\cos x$:

$$\sin^3 x = (\sin^2 x)(\sin x) = (1 - \cos^2 x) \sin x.$$

Now substitute $u = \cos x$, $du = -\sin x \, dx$:

$$\int \sin^3 x \, dx = \int (1 - \cos^2 x) \sin x \, dx = - \int (1 - u^2) \, du = \frac{u^3}{3} - u + C = \frac{\cos^3 x}{3} - \cos x + C.$$

This strategy works whenever $\sin^m x$ occurs with m odd, whatever power of $\cos x$ is present. If instead n is odd, we split off one factor of $\cos x$ and write the remaining even power of $\cos x$ as a power of $1 - \sin^2 x$.

Example 16.2 (Odd power of $\cos x$). Evaluate $\int \sin^2 x \cos^3 x \, dx$.

Solution. Here $\cos^3 x$ is the odd power, so

$$\sin^2 x \cos^3 x = \sin^2 x (\cos^2 x)(\cos x) = \sin^2 x (1 - \sin^2 x)(\cos x),$$

and the substitution $u = \sin x$, $du = \cos x \, dx$ gives

$$\int \sin^2 x \cos^3 x \, dx = \int (u^2 - u^4) \, du = \frac{u^3}{3} - \frac{u^5}{5} + C = \frac{\sin^3 x}{3} - \frac{\sin^5 x}{5} + C.$$

A different strategy is needed when neither $\sin x$ nor $\cos x$ occurs to an odd power. In that case we fall back on the double-angle identities (13.3) of Section 13,

$$\sin^2 x = \frac{1}{2}(1 - \cos 2x), \quad \cos^2 x = \frac{1}{2}(1 + \cos 2x),$$

from which we computed there that

$$\int \sin^2 x \, dx = \frac{x}{2} - \frac{\sin 2x}{4} + C, \quad \int \cos^2 x \, dx = \frac{x}{2} + \frac{\sin 2x}{4} + C. \quad (16.1)$$

Since $\sin 2x = 2 \sin x \cos x$, these two answers may also be written as $\frac{x}{2} \mp \frac{1}{2} \sin x \cos x + C$.

Example 16.3. Evaluate $\int \sin^4 x \, dx$.

Solution. Apply the identity for $\sin^2 x$, and then apply it a second time to the $\cos^2 2x$ that appears:

$$\begin{aligned} \int \sin^4 x \, dx &= \int (\sin^2 x)^2 \, dx = \int \left(\frac{1}{2}(1 - \cos 2x)\right)^2 \, dx = \frac{1}{4} \int (1 - 2 \cos 2x + \cos^2 2x) \, dx \\ &= \frac{1}{4} \int \left(1 - 2 \cos 2x + \frac{1 + \cos 4x}{2}\right) \, dx \\ &= \frac{1}{4} \int \left(\frac{3}{2} - 2 \cos 2x + \frac{\cos 4x}{2}\right) \, dx. \end{aligned}$$

Simple substitutions now give

$$\int \sin^4 x \, dx = \frac{1}{4} \left(\frac{3x}{2} - \sin 2x + \frac{\sin 4x}{8} \right) + C = \frac{3x}{8} - \frac{\sin 2x}{4} + \frac{\sin 4x}{32} + C.$$

Method (Integrating $\sin^m x \cos^n x$).

- (1) m odd: split off one power of $\sin x$ and use $\sin^2 x = 1 - \cos^2 x$ to express the remaining powers of $\sin x$ in terms of $\cos x$; then substitute $u = \cos x$, $du = -\sin x \, dx$.
- (2) n odd: split off one power of $\cos x$ and use $\cos^2 x = 1 - \sin^2 x$ to express the remaining powers of $\cos x$ in terms of $\sin x$; then substitute $u = \sin x$, $du = \cos x \, dx$.

- (3) m and n both even: either use the double-angle identities repeatedly, or convert the integrand into an expression entirely in $\sin x$ or entirely in $\cos x$ and apply a reduction formula.

The reduction formula (11.6) for $\int \sin^n x \, dx$ has a companion for $\int \cos^n x \, dx$, derived in the same way:

$$\int \cos^n x \, dx = \frac{\cos^{n-1} x \sin x}{n} + \frac{n-1}{n} \int \cos^{n-2} x \, dx. \quad (16.2)$$

Example 16.4 (Even power of $\cos x$). Evaluate $\int \cos^4 x \, dx$.

Solution. Apply (16.2) with $n = 4$, and then (16.1) in the form $\int \cos^2 x \, dx = \frac{1}{2} \cos x \sin x + \frac{1}{2}x + C$:

$$\begin{aligned} \int \cos^4 x \, dx &= \frac{1}{4} \cos^3 x \sin x + \frac{3}{4} \int \cos^2 x \, dx \\ &= \frac{1}{4} \cos^3 x \sin x + \frac{3}{4} \left(\frac{1}{2} \cos x \sin x + \frac{1}{2}x \right) + C \\ &= \frac{1}{4} \cos^3 x \sin x + \frac{3}{8} \cos x \sin x + \frac{3}{8}x + C. \end{aligned}$$

Comment. Different methods may produce antiderivatives that look quite different. This is no contradiction: if $\int f(x) \, dx$ equals both $F(x) + C$ and $G(x) + C$, it does not follow that $F = G$, only that F and G differ by a constant. For instance the substitutions $u = \sin x$ and $u = \cos x$ give

$$\int 2 \sin x \cos x \, dx = \sin^2 x + C \quad \text{and} \quad \int 2 \sin x \cos x \, dx = -\cos^2 x + C,$$

and the two agree because $\sin^2 x = -\cos^2 x + 1$, the extra 1 being absorbed into the arbitrary constant.

16.1 Secants and tangents

Of the six standard trigonometric functions, $\sin x$ and $\cos x$ have antiderivatives read off from their derivative formulas, and $\int \tan x \, dx = \ln |\sec x| + C$ follows from the substitution $u = \cos x$ (see §5.7 of the textbook); the integral of $\cot x$ is handled the same way. That leaves $\sec x$ and $\csc x$.

Example 16.5 (Integral of the secant). Derive the formula

$$\int \sec x \, dx = \ln |\sec x + \tan x| + C. \quad (16.3)$$

Solution. The trick is to multiply the integrand by an unusual fraction equal to 1, chosen so that a substitution becomes available:

$$\int \sec x \, dx = \int \sec x \left(\frac{\sec x + \tan x}{\sec x + \tan x} \right) dx = \int \frac{\sec^2 x + \sec x \tan x}{\sec x + \tan x} dx.$$

The numerator is exactly the derivative of the denominator, so with $u = \sec x + \tan x$ and $du = (\sec x \tan x + \sec^2 x) dx$ the integral becomes

$$\int \frac{du}{u} = \ln |u| + C = \ln |\sec x + \tan x| + C.$$

The same method applied to $\csc x$ gives $\int \csc x dx = \ln |\csc x - \cot x| + C$.

Method (Integrating $\tan^m x \sec^n x$).

- (1) m odd and $n \geq 1$: separate out the factor $\sec x \tan x$ and use $\tan^2 x = \sec^2 x - 1$ to express the rest of the integrand in terms of $\sec x$; then substitute $u = \sec x$, $du = \sec x \tan x dx$.
- (2) n even: separate out a factor of $\sec^2 x$ and use $\sec^2 x = 1 + \tan^2 x$ to express the rest of the integrand in terms of $\tan x$; then substitute $u = \tan x$, $du = \sec^2 x dx$.
- (3) m even and n odd: use $\tan^2 x = \sec^2 x - 1$ to obtain an integral involving only powers of $\sec x$, and apply the reduction formula (16.4) below.

In the first two cases the substitution leaves an integral in powers of u alone.

Example 16.6. Evaluate $\int \tan^3 x \sec^3 x dx$.

Solution. The integrand contains a copy of $\sec x \tan x$, and the remaining powers of $\tan x$ are even, so we are in the first case. Using $\tan^2 x = \sec^2 x - 1$,

$$\int \tan^3 x \sec^3 x dx = \int (\sec^2 x - 1)(\sec^2 x)(\sec x \tan x) dx = \int (\sec^4 x - \sec^2 x) \sec x \tan x dx,$$

and with $u = \sec x$, $du = \sec x \tan x dx$,

$$\int \tan^3 x \sec^3 x dx = \int (u^4 - u^2) du = \frac{u^5}{5} - \frac{u^3}{3} + C = \frac{\sec^5 x}{5} - \frac{\sec^3 x}{3} + C.$$

Example 16.7. Evaluate $\int \tan^4 x \sec^4 x dx$.

Solution. Now $\sec x$ occurs to an even power. Since the derivative of $\tan x$ is $\sec^2 x$, we separate out $\sec^2 x$ and convert the rest into powers of $\tan x$ using $\sec^2 x = \tan^2 x + 1$:

$$\int \tan^4 x \sec^4 x dx = \int (\tan^4 x)(\tan^2 x + 1)(\sec^2 x) dx = \int (\tan^6 x + \tan^4 x) \sec^2 x dx.$$

Setting $u = \tan x$, $du = \sec^2 x dx$,

$$\int \tan^4 x \sec^4 x dx = \int (u^6 + u^4) du = \frac{\tan^7 x}{7} + \frac{\tan^5 x}{5} + C.$$

16.2 A table of trigonometric integrals

The following formulas collect what we have derived, together with some related ones obtained by the same methods. In the last three, $m \neq \pm n$.

$$\begin{aligned}
 \int \tan x \, dx &= \ln |\sec x| + C = -\ln |\cos x| + C, \\
 \int \cot x \, dx &= -\ln |\csc x| + C = \ln |\sin x| + C, \\
 \int \csc x \, dx &= \ln |\csc x - \cot x| + C, \\
 \int \tan^m x \, dx &= \frac{\tan^{m-1} x}{m-1} - \int \tan^{m-2} x \, dx, \\
 \int \cot^m x \, dx &= -\frac{\cot^{m-1} x}{m-1} - \int \cot^{m-2} x \, dx, \\
 \int \sec^m x \, dx &= \frac{\tan x \sec^{m-2} x}{m-1} + \frac{m-2}{m-1} \int \sec^{m-2} x \, dx, \\
 \int \csc^m x \, dx &= -\frac{\cot x \csc^{m-2} x}{m-1} + \frac{m-2}{m-1} \int \csc^{m-2} x \, dx,
 \end{aligned} \tag{16.4}$$

$$\int \sin mx \sin nx \, dx = \frac{\sin(m-n)x}{2(m-n)} - \frac{\sin(m+n)x}{2(m+n)} + C, \tag{16.5}$$

$$\int \sin mx \cos nx \, dx = -\frac{\cos(m-n)x}{2(m-n)} - \frac{\cos(m+n)x}{2(m+n)} + C, \tag{16.6}$$

$$\int \cos mx \cos nx \, dx = \frac{\sin(m-n)x}{2(m-n)} + \frac{\sin(m+n)x}{2(m+n)} + C. \tag{16.7}$$

The integrals (16.5)–(16.7) of the products $\sin mx \sin nx$, $\sin mx \cos nx$, and $\cos mx \cos nx$ appear in the theory of Fourier series, a technique used extensively in engineering and physics.

Example 16.8. Evaluate $\int_0^\pi \sin 5x \cos 2x \, dx$.

Solution. Apply (16.6) with $m = 5$ and $n = 2$:

$$\int_0^\pi \sin 5x \cos 2x \, dx = \left(-\frac{\cos 3x}{6} - \frac{\cos 7x}{14} \right) \Big|_0^\pi = \left(\frac{1}{6} + \frac{1}{14} \right) - \left(-\frac{1}{6} - \frac{1}{14} \right) = \frac{10}{21}.$$

17 Integrals of hyperbolic functions

This section covers §7.4 of the textbook. The hyperbolic functions resemble the trigonometric functions, and their derivative formulas resemble the trigonometric ones up to at most a sign. Read backwards, those derivative formulas are integral formulas.

Method (Hyperbolic integral formulas).

$$\begin{aligned}\int \sinh x \, dx &= \cosh x + C, & \int \cosh x \, dx &= \sinh x + C, \\ \int \operatorname{sech}^2 x \, dx &= \tanh x + C, & \int \operatorname{csch}^2 x \, dx &= -\coth x + C, \\ \int \operatorname{sech} x \tanh x \, dx &= -\operatorname{sech} x + C, & \int \operatorname{csch} x \coth x \, dx &= -\operatorname{csch} x + C.\end{aligned}$$

Remark 17.1. None of this needs new machinery. The hyperbolic functions are built from the real exponential,

$$\sinh x = \frac{e^x - e^{-x}}{2}, \quad \cosh x = \frac{e^x + e^{-x}}{2},$$

so any one of these integrals can also be computed by writing the integrand in terms of e^x and e^{-x} . Compare the exponential form (15.4) of the sine and cosine: the trigonometric functions need the complex exponential, the hyperbolic functions only the real one. This is why hyperbolic integrals go quickly.

Example 17.2. Calculate $\int x \sinh(x^2) \, dx$.

Solution. The substitution $u = x^2$, $du = 2x \, dx$ yields

$$\int x \sinh(x^2) \, dx = \frac{1}{2} \int \sinh u \, du = \frac{1}{2} \cosh u + C = \frac{1}{2} \cosh(x^2) + C.$$

The techniques of Section 16 carry over to hyperbolic integrals with little change: in place of the trigonometric identities we use the corresponding hyperbolic ones,

$$\begin{aligned}\cosh^2 x - \sinh^2 x &= 1, & \cosh^2 x &= 1 + \sinh^2 x, & \sinh 2x &= 2 \sinh x \cosh x, \\ \cosh^2 x &= \frac{1}{2}(\cosh 2x + 1), & \sinh^2 x &= \frac{1}{2}(\cosh 2x - 1), & \cosh 2x &= \cosh^2 x + \sinh^2 x.\end{aligned}\tag{17.1}$$

Caution. The signs in (17.1) are not the trigonometric ones. Note in particular that $\sinh^2 x = \frac{1}{2}(\cosh 2x - 1)$, whereas $\sin^2 x = \frac{1}{2}(1 - \cos 2x)$.

Example 17.3 (Powers of $\sinh x$ and $\cosh x$). Calculate

(a) $\int \sinh^3 x \cosh^2 x \, dx$ and

(b) $\int \cosh^2 x \, dx$.

Solution.

- (a) Since $\sinh x$ appears to an odd power, split off one factor of it and use $\sinh^2 x = \cosh^2 x - 1$:

$$\sinh^3 x \cosh^2 x = (\cosh^2 x - 1)(\cosh^2 x) \sinh x.$$

Then substitute $u = \cosh x$, $du = \sinh x \, dx$:

$$\int \sinh^3 x \cosh^2 x \, dx = \int (u^2 - 1)u^2 \, du = \frac{u^5}{5} - \frac{u^3}{3} + C = \frac{\cosh^5 x}{5} - \frac{\cosh^3 x}{3} + C.$$

(b) Here the powers are even, so we use the identity $\cosh^2 x = \frac{1}{2}(\cosh 2x + 1)$:

$$\int \cosh^2 x \, dx = \frac{1}{2} \int (\cosh 2x + 1) \, dx = \frac{1}{2} \left(\frac{\sinh 2x}{2} + x \right) + C = \frac{1}{4} \sinh 2x + \frac{1}{2} x + C.$$

Remark 17.4. Hyperbolic substitution is an alternative to the trigonometric substitution of §7.3 of the textbook for integrands containing the square root expressions below. The identities (17.1) are what make the square roots disappear.

$$\begin{aligned} \text{for } \sqrt{x^2 + a^2} : \quad x &= a \sinh u, & dx &= a \cosh u \, du, & \sqrt{x^2 + a^2} &= a \cosh u, \\ \text{for } \sqrt{x^2 - a^2} : \quad x &= a \cosh u, & dx &= a \sinh u \, du, & \sqrt{x^2 - a^2} &= a \sinh u. \end{aligned}$$

18 Integration of functions which take complex values

This section covers §3 of the supplement, which supplements §§7.2 and 7.4 of the textbook.

A function such as $f(x) = (1 + 2i)x + 3ix^2$ may take complex values, but its variable x is only allowed to take on real values, and it is only for functions of this kind that we define definite integrals. In this case nothing differs from what we already learned about integrals of real-valued functions:

- the Riemann sum definition of an integral still applies;
- the fundamental theorem of calculus is still true;
- the properties of integrals, substitution and integration by parts included, still work;
- the formulas we have derived, and those in the table at the back of the textbook, are all still valid when the complex functions are defined appropriately, as we did for e^{a+bi} in (15.2) — with one important exception, in the caution below.

Caution. All absolute values must be removed from logarithms. For example,

$$\int \frac{dx}{x+i} = \ln(x+i) + C, \quad \text{and not} \quad \ln|x+i| + C.$$

One can see that $\ln|x+i|$ will not work because it is always a real number, so its derivative cannot be the complex number $\frac{1}{x+i}$.¹

Example 18.1. Compute $\int_0^2 [(1 + 2i)x + 3ix^2] \, dx$.

Solution. Split the integral by linearity and integrate as usual:

$$\int_0^2 [(1 + 2i)x + 3ix^2] \, dx = (1 + 2i) \int_0^2 x \, dx + i \int_0^2 3x^2 \, dx = \left(\frac{(1 + 2i)x^2}{2} + ix^3 \right) \Big|_0^2,$$

which is $(1 + 2i)2 + 8i - 0 = 2 + 12i$.

¹How, then, is the logarithm of a complex number defined? We do not discuss that here, but if you want to think about it, here is a hint: remember that the logarithm is the inverse of the exponential, and that $x + iy = re^{i\theta}$.

18.1 Integrating products of sines, cosines, and exponentials

In Section 16 products of sines and cosines were integrated using trigonometric identities. There are other ways to do this now that we have complex exponentials: by (15.4), every such product is a combination of complex exponentials, and exponentials are easy to integrate. Examples will make this clearer.

Example 18.2. Let us integrate $8 \cos(3x) \sin(x)$.

Solution. First rewrite the integrand using (15.4):

$$8 \cos(3x) \sin(x) = 8 \left(\frac{e^{3ix} + e^{-3ix}}{2} \right) \left(\frac{e^{ix} - e^{-ix}}{2i} \right) = \frac{2}{i} (e^{4ix} + e^{-2ix} - e^{2ix} - e^{-4ix}).$$

Integrating, and then grouping the terms in pairs so as to recognize cosines,

$$\begin{aligned} \int 8 \cos(3x) \sin(x) dx &= \frac{2}{i} \int (e^{4ix} + e^{-2ix} - e^{2ix} - e^{-4ix}) dx \\ &= \frac{2}{i} \left[\frac{e^{4ix}}{4i} - \frac{e^{-2ix}}{2i} - \frac{e^{2ix}}{2i} + \frac{e^{-4ix}}{4i} \right] + C \\ &= \frac{2}{i} \left[\frac{e^{4ix}}{4i} + \frac{e^{-4ix}}{4i} - \frac{e^{-2ix}}{2i} - \frac{e^{2ix}}{2i} \right] + C \\ &= - \left(\frac{e^{4ix} + e^{-4ix}}{2} \right) + 2 \left(\frac{e^{2ix} + e^{-2ix}}{2} \right) + C \\ &= -\cos(4x) + 2\cos(2x) + C. \end{aligned}$$

Example 18.3. Let us integrate $e^{2x} \sin(x)$.

Solution. Problems like this were solved in Section 11 by using integration by parts twice. Here is another way. Using the formula for the sine and integrating,

$$\begin{aligned} \int e^{2x} \sin(x) dx &= \frac{1}{2i} \int e^{2x} (e^{ix} - e^{-ix}) dx = \frac{1}{2i} \int (e^{(2+i)x} - e^{(2-i)x}) dx \\ &= \frac{1}{2i} \left(\frac{e^{(2+i)x}}{2+i} - \frac{e^{(2-i)x}}{2-i} \right) + C = -\frac{e^{2x}}{2} \left(\frac{e^{ix}}{1-2i} + \frac{e^{-ix}}{1+2i} \right) + C, \end{aligned}$$

the last step using $2i(2+i) = -2(1-2i)$ and $2i(2-i) = 2(1+2i)$. Clearing the complex numbers from the denominators and regrouping,

$$\begin{aligned} \int e^{2x} \sin(x) dx &= -\frac{e^{2x}}{2} \left[\frac{(1+2i)e^{ix}}{5} + \frac{(1-2i)e^{-ix}}{5} \right] + C \\ &= -\frac{e^{2x}}{5} \left[\left(\frac{e^{ix} + e^{-ix}}{2} \right) - 2 \left(\frac{e^{ix} - e^{-ix}}{2i} \right) \right] + C \\ &= -\frac{e^{2x}}{5} [\cos(x) - 2\sin(x)] + C. \end{aligned}$$

Comment. This method works for integrals of products of sines, cosines, and exponentials, and often for quotients of them as well, though those require more advanced methods such as partial fractions. The advantage of using complex exponentials is that it is often simpler than the other approaches, although at first it feels algebraically cumbersome.