

# MATH 109 – Mathematical Reasoning

## Homework 7: Counting subsets and infinite sets

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Due December 2, 2025

1. Find the coefficient of  $a^{98}b^2$  in  $(a + b)^{100}$ .
2. By equating the coefficients of  $x^n$  in  $(1 + x)^{2n} = (1 + x)^n(1 + x)^n$ , prove that

$$\binom{2n}{n} = \sum_{r=0}^n \binom{n}{r}^2.$$

3. Let  $X$  and  $Y$  be disjoint sets and let  $r \in \mathbb{N}$ .

(a) Prove that the function

$$f: \bigcup_{k=0}^r \mathcal{P}_k(X) \times \mathcal{P}_{r-k}(Y) \rightarrow \mathcal{P}_r(X \cup Y)$$
$$(A, B) \mapsto A \cup B$$

is bijective.

(b) Use part (a) to deduce that

$$\binom{n+m}{r} = \sum_{k=0}^r \binom{n}{k} \binom{m}{r-k}.$$

[**Hint:** Let  $n = |X|$  and  $m = |Y|$  in part (a) and use the fact that if  $S_1$  and  $S_2$  are two finite sets then  $|S_1 \times S_2| = |S_1| \cdot |S_2|$ .]

4. (a) Let  $A$  and  $B$  be enumerable sets. Prove that  $A \cup B$  is enumerable.  
(b) Let  $A$  be an enumerable subset of an uncountable set  $X$ . Prove that  $X \setminus A$  is uncountable. [**Hint:** Proof by contradiction using part (a).]
5. Let  $X$  and  $Y$  be two (possibly infinite) sets with  $|X| = |Y|$ . Prove that  $|\mathcal{P}(X)| = |\mathcal{P}(Y)|$ .
6. Let  $a, b \in \mathbb{R}$  with  $a < b$ . Prove that  $|(a, b)| = |(0, 1)|$ . [**Hint:** Construct a bijective function  $f: (a, b) \rightarrow (0, 1)$ .]

## Practice exercises

The following exercises are for your practice only and will not be graded. The exercises use HaLLMos, a website where you can write proofs in natural language, receive immediate feedback, and revise and resubmit as often as you like.

1. Do the following problem set on HaLLMos: Q1, Q2, Q3.
2. Use the Sandbox feature of HaLLMos to check and improve on your answers to the homework problems. This  $\text{\LaTeX}$  guide may be helpful!
3. Do some exercises from Chapters 12 and 14 of the textbook. Then use the Sandbox feature of HaLLMos to check your answers.