

MATH 109 – Mathematical Reasoning

Homework 5: Functions

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Due November 18, 2025

1. Define functions $f, g: \mathbb{R} \rightarrow \mathbb{R}$ by

$$f(x) = x^3, \quad g(x) = 1 - x.$$

- (a) Find the functions

- i. $f \circ f$,
- ii. $f \circ g$,
- iii. $g \circ f$,
- iv. $g \circ g$.

- (b) List the elements of the set $\{x \in \mathbb{R} \mid fg(x) = gf(x)\}$.

2. Let $f: A \rightarrow B$ and $g: C \rightarrow D$ be functions. Define a new function

$$f \times g: A \times C \rightarrow B \times D$$
$$(a, c) \mapsto (f(a), g(c)).$$

Prove or disprove the following.

- (a) If f and g are injective, then $f \times g$ is injective.
 - (b) If f and g are surjective, then $f \times g$ is surjective.
3. Prove that a composition of surjections is a surjection and that a composition of injections is an injection.
4. Let $f: \mathbb{R} \rightarrow \mathbb{R}$ be a function. Given $x \in \mathbb{R}$, the function f is said to be *continuous* at x if for all $\varepsilon > 0$, there exists $\delta > 0$ such that for any $y \in \mathbb{R}$ with $|x - y| < \delta$, we have

$$|f(x) - f(y)| < \varepsilon.$$

The function f is said to be *continuous* if it is continuous at every $x \in \mathbb{R}$.

Define a function

$$f: \mathbb{R} \rightarrow \mathbb{R}$$
$$x \mapsto \begin{cases} 0, & \text{if } x \leq 0, \\ 1, & \text{if } x > 0. \end{cases}$$

Prove or disprove: the function f is continuous.

5. Let $f: \mathbb{R} \rightarrow \mathbb{R}$ be a function from the real numbers to itself. The function f is called *uniformly continuous* if for all $\varepsilon > 0$, there exists $\delta > 0$ such that for any pair of real numbers x, y with $|x - y| < \delta$, we have that $|f(x) - f(y)| < \varepsilon$.

What does it mean for a function $f: \mathbb{R} \rightarrow \mathbb{R}$ to **not** be uniformly continuous? [**Hint:** Observe that you do not need to “know” anything about uniform continuity to answer this question.]

6. Let $f: X \rightarrow Y$ be a function. Prove that f is surjective if and only if there exists a function $g: Y \rightarrow X$ such that $f \circ g = \text{id}_Y$. Is g necessarily unique in this case?

Practice exercises

The following exercises are for your practice only and will not be graded. The exercises use HaLLMos, a website where you can write proofs in natural language, receive immediate feedback, and revise and resubmit as often as you like.

1. Do problem set 7 (functions) on HaLLMos.
2. Use the Sandbox feature of HaLLMos to check and improve on your answers to the homework problems. This L^AT_EXguide may be helpful!