

MATH 109 – Mathematical Reasoning

Homework 4: Quantifiers

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- The following are standard subsets of the set of real numbers known as the *real intervals* with endpoints the real numbers a and b .
 - The *open interval* $(a, b) := \{x \in \mathbb{R} \mid a < x < b\}$.
 - The *closed interval* $[a, b] := \{x \in \mathbb{R} \mid a \leq x \leq b\}$.
 - The *right half-open interval* $[a, b) := \{x \in \mathbb{R} \mid a \leq x < b\}$.
 - The *left half-open interval* $(a, b] := \{x \in \mathbb{R} \mid a < x \leq b\}$.
 - Find the elements of the set $[a, b] \setminus (a, b)$.
 - Prove that $(a, b) = \emptyset$ if and only if $a \geq b$. [**Hint:** Prove the contrapositive.]
 - Prove that if $a \leq b$, then $[a, b] \subseteq (c, d)$ if and only if $c < a$ and $b < d$.
- Prove the following equalities of sets.
 - $\{x \in \mathbb{R} \mid x^2 + x - 2 = 0\} = \{1, -2\}$.
 - $\{x \in \mathbb{R} \mid x^2 + x - 2 < 0\} = (-2, 1)$.
 - $\{x \in \mathbb{R} \mid x^2 + x - 2 > 0\} = (-\infty, -2) \cup (1, \infty)$.

In part (c), the infinite intervals are defined as follows:

$$\begin{aligned}(-\infty, -2) &:= \{x \in \mathbb{R} \mid x < -2\}, \\(1, \infty) &:= \{x \in \mathbb{R} \mid x > 1\}.\end{aligned}$$

- Prove or disprove each of the following statements. [**Hint:** to disprove a statement, you must prove its negation.]
 - $\forall x \in \mathbb{R}, \exists y \in \mathbb{R}: x + y > 0$.
 - $\forall x \in \mathbb{R}, \exists y \in \mathbb{R}: x - y > 0$.
 - $\exists x \in \mathbb{R}: \forall y \in \mathbb{R}, x + y > 0$.
 - $\forall x \in \mathbb{R}, \exists y \in \mathbb{R}: xy > 0$.
 - $\exists x \in \mathbb{R}: \forall y \in \mathbb{R}, xy > 0$.
 - $\forall x \in \mathbb{R}, \exists y \in \mathbb{R}: x + y \geq 0$.
 - $\exists x \in \mathbb{R}: \forall y \in \mathbb{R}, xy \geq 0$.
 - $\forall x \in \mathbb{R}, \exists y \in \mathbb{R}: (x + y > 0 \text{ or } x + y = 0)$.

- (i) $\forall x \in \mathbb{R}, \exists y \in \mathbb{R}: (x + y > 0 \text{ and } x + y = 0)$.
 - (j) $(\forall x \in \mathbb{R}, \exists y \in \mathbb{R}: x + y > 0) \text{ and } (\forall x \in \mathbb{R}, \exists y \in \mathbb{R}: x + y = 0)$.
4. Suppose that $A \subseteq \mathbb{Z}$.
- (a) Write out the negation of the following statement in symbols: “There is a greatest number in A ”.
 - (b) Give an example of a set A for which the statement in part (a) is true, and another example for which it is false.

Practice exercises

The following exercises are for your practice only and will not be graded. The exercises use HaLLMos, a website where you can write proofs in natural language, receive immediate feedback, and revise and resubmit as often as you like.

1. Do problem set 5 (sets) on HaLLMos.
2. Use the Sandbox feature of HaLLMos to check and improve on your answers to the homework problems. This L^AT_EX guide may be helpful!
3. Do some exercises from Chapters 6 and 7 of the textbook. Then use the Sandbox feature of HaLLMos to check your answers.