

# MATH 109 – Mathematical Reasoning

## Homework 2: Proofs

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1. Let  $x$  and  $y$  be *non-positive* real numbers. Using only the inequality axioms, prove that  $x < y$  implies  $x^2 > y^2$ . We recall the inequality axioms below.

1. **Trichotomy law:** For each pair of real numbers  $a, b$ , one and only one of the three possibilities  $a < b$ ,  $a = b$ ,  $a > b$  is true.
2. **Addition law:** For real numbers  $a, b, c$ ,

$$a < b \iff a + c < b + c.$$

3. **Multiplication law:** For real numbers  $a, b, c$ ,

$$\begin{aligned} a < b &\iff ac < bc \quad (\text{if } c > 0), \\ a < b &\iff ac > bc \quad (\text{if } c < 0). \end{aligned}$$

4. **Transitive law:** For real numbers  $a, b, c$ ,

$$(a < b \text{ and } b < c) \implies a < c.$$

2. We know from class that  $(a < b) \implies 4ab < (a + b)^2$ . Is it true that

$$(a < b) \iff 4ab < (a + b)^2?$$

If so, prove it.

3. What is wrong with the following “proof” that 1 is the largest integer? What does this argument actually show?

*Proof.* Let  $n$  be the largest integer. Since 1 is an integer we must have  $1 \leq n$ . Since  $n^2$  is also an integer we must have  $n^2 \leq n$ . Since  $n$  is positive, the inequality  $n^2 \leq n$  implies  $n \leq 1$ . Since  $1 \leq n$  and  $n \leq 1$ , we have that  $n = 1$ , as desired.  $\square$

4. Prove by contradiction that there do not exist integers  $n$  and  $m$  such that  $14n + 21m = 100$ .
5. Prove that for any integer  $n$ ,

$$n^2 \text{ is odd} \implies n \text{ is odd}$$

- (a) by contradiction.
  - (b) by writing down its contrapositive.
6. Prove by contradiction from the trichotomy law that, for any real numbers  $a, b$ ,

$$(a \leq b \text{ and } b \leq a) \implies a = b.$$

7. Prove by contradiction that there does not exist a largest integer. [**Hint:** Observe that for any integer  $n$ , there is a greater one, say  $n + 1$ . You may want to begin your proof as follows: “Assume, for a contradiction, that there is a largest integer. Call this integer  $n$ ...”]