

MATH 103B – Modern Algebra II

Homework 7

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Due March 10

1. Let V be a vector space over a field $\mathbb{F}$ and let B be a basis for V . Prove that every element of V is a **unique** linear combination of elements of B . In other words, prove that if

$$v = a_1v_1 + \cdots + a_nv_n = b_1v_1 + \cdots + b_nv_n$$

for some $a_i, b_i \in \mathbb{F}$ and $v_i \in B$, then $a_i = b_i$ for all i .

2. By definition, a *traceless* matrix is a matrix whose trace is zero. For example, a 2×2 matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is traceless if $a + d = 0$. Let $S := \{M \in M_2(\mathbb{R}) \mid M \text{ is traceless}\}$ be the set of 2×2 traceless matrices over $\mathbb{R}$.

(a) Show that S is a subspace of $M_2(\mathbb{R})$.

(b) Find a basis of S over $\mathbb{R}$.

(c) What is the dimension of the vector space S over $\mathbb{R}$?

3. (a) Show that the set

$$B := \{1, x, x^2, x^3, \dots\}$$

is a basis for $\mathbb{R}[x]$.

(b) Using part (a), or otherwise, show that $\mathbb{R}[x]$ an infinite-dimensional vector space over $\mathbb{R}$. [**Hint:** Use Theorem 11.16 from the lecture notes.]

4. Show that

$$\mathbb{Q}[\sqrt[3]{2}] := \{a + b\sqrt[3]{2} + c\sqrt[3]{4} \mid a, b, c \in \mathbb{Q}\}$$

is a field extension of $\mathbb{Q}$ of degree 3. [**Hint:** To show that $\mathbb{Q}[\sqrt[3]{2}]$ is a field, you can show that

$$\mathbb{Q}[\sqrt[3]{2}] \cong \frac{\mathbb{Q}[x]}{(x^3 - 2)}$$

using the first isomorphism theorem and use Theorem 6.3 and Corollary 9.20 from the lecture notes. You may use without proof that if $f(x) \in \mathbb{Q}[x]$ with $f(\sqrt[3]{2}) = 0$, then $x^3 - 2$ divides f .]

5. Define

$$\mathbb{F}_{16} := \frac{\mathbb{Z}_2[x]}{(x^4 + x + \bar{1})}$$

(you do not have to prove that $\mathbb{F}_{16}$ is a field, which would involve proving that $x^4 + x + \bar{1}$ is irreducible). Define

$$y := x + (x^4 + x + \bar{1}) \in \mathbb{F}_{16}.$$

- (a) What is the minimal polynomial of y over $\mathbb{Z}_2$?
- (b) What is the minimal polynomial of $y + 1$ over $\mathbb{Z}_2$?
- (c) What is the minimal polynomial of $y^2 + y$ over $\mathbb{Z}_2$?