

# MATH 103B – Modern Algebra II

## Homework 6

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Due March 3

1. Let  $R$  be a UFD and let  $r \in R$  be an irreducible element. Prove that  $r$  is prime. [Hint: Assuming  $r$  divides  $st$ , use the uniqueness of the factorization of  $s$  and  $t$  into irreducibles.]

2. Let  $R$  be a ring (not necessarily commutative) and let

$$I_1 \subseteq I_2 \subseteq I_3 \subseteq \dots$$

be an ascending chain of ideals of  $R$ . Show that

$$I := \bigcup_{k=1}^{\infty} I_k$$

is an ideal of  $R$ .

3. Let  $R$  be an integral domain, and let  $r, s \in R \setminus \{0\}$  be non-unit elements of  $R$ . Prove that  $(rs) \subsetneq (r)$  (in other words,  $(rs)$  is a subset of  $(r)$ , and  $(rs) \neq (r)$ ).
4. Prove that the ring

$$\mathbb{Z}[\sqrt{2}] = \{a + b\sqrt{2} \mid a, b \in \mathbb{Z}\}$$

is a Euclidean domain with

$$d(a + b\sqrt{2}) = |a^2 - 2b^2|.$$

[Hint: Try to emulate the proof of Proposition 10.15 from the lecture notes.]

5. Consider the ring of Gaussian integers  $\mathbb{Z}[i] = \{a + bi \mid a, b \in \mathbb{Z}\}$ , and define

$$\begin{aligned} d: \mathbb{Z}[i] &\rightarrow \mathbb{N} \\ a + bi &\mapsto a^2 + b^2. \end{aligned}$$

In this question, you may use that  $d(z_1 z_2) = d(z_1) d(z_2)$  for all  $z_1, z_2 \in \mathbb{Z}[i]$  (recall that this was proved in the proof of Proposition 10.15 from the lecture notes).

- (a) Show that the units of  $\mathbb{Z}[i]$  are  $\pm 1$  and  $\pm i$ .
- (b) Show that if  $a^2 + b^2$  is a prime number, then  $a + bi$  is a prime element of  $\mathbb{Z}[i]$ .
- (c) Show that 2, 5, and 13 are **not** prime elements of  $\mathbb{Z}[i]$  by finding their prime factorizations.

6. Come up with examples of the following.

- (a) A ring which is a Euclidean domain but not a field.
- (b) A ring which is a PID but not a Euclidean domain.
- (c) A ring which is a UFD but not a PID.
- (d) A ring which is an integral domain but not a UFD.
- (e) A commutative ring which is not an integral domain.
- (f) A noncommutative ring.

You do not need to produce detailed proofs for each part, but you should briefly explain why your ring satisfies the desired properties.