

MATH 103B – Modern Algebra II

Homework 5

Lucas Buzaglo

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Throughout, the notation $\mathbb{F}$ is used for an arbitrary field.

1. Let $a \in \mathbb{F}$ and $f(x) \in \mathbb{F}[x]$. Use the division theorem for polynomial rings to prove the following.
 - (a) $f(a)$ is the remainder in the division of $f(x)$ by the polynomial $x - a$.
 - (b) a is a zero of $f(x)$ if and only if $x - a$ divides $f(x)$.
2.
 - (a) Prove that the polynomial $x^3 - x$ has more than three roots in $\mathbb{Z}_6$. [**Hint:** $\mathbb{Z}_6$ only has 6 elements, so you can do this by brute force.]
 - (b) Explain why this does not contradict the fact that a polynomial in $\mathbb{F}[x]$ of degree n has at most n zeros (this is Corollary 7.8 from the lecture notes).
3.
 - (a) Prove that $\mathbb{F}[x]^* = \mathbb{F}^* = \mathbb{F} \setminus \{0\}$. In other words, show that if $f \in \mathbb{F}[x]$ is a unit, then f is a nonzero constant polynomial (does not have any x 's).
 - (b) Using part (a), or otherwise, deduce that a polynomial $f \in \mathbb{F}[x]$ is reducible if and only if it can be expressed as a product of two non-constant polynomials.
 - (c) Calculate $(\bar{3}x + \bar{1})^3$ in $\mathbb{Z}_9[x]$. Is $\bar{3}x + \bar{1}$ a unit?
 - (d) Explain why your findings from part (c) do not contradict part (a).
4. Using Gauss's lemma, or otherwise, prove that

$$c(fg) = c(f)c(g)$$

for all $f, g \in \mathbb{Z}[x]$.

5. Define

$$\mathbb{F}_9 := \frac{\mathbb{Z}_3[x]}{(x^2 + \bar{1})}.$$

- (a) Show that $\mathbb{F}_9$ is a field with 9 elements.
 - (b) Let $y := x + (x^2 + \bar{1}) \in \mathbb{F}_9$ be the coset represented by x in $\mathbb{F}_9$. Compute the following products in $\mathbb{F}_9$, simplifying as much as possible.
 - i. $(y + \bar{1})^2$.
 - ii. $(y^3 + \bar{2}y + \bar{1})(y^2 + \bar{2})$.
6. List all irreducible polynomials in $\mathbb{Z}_2[x]$ of degrees 2 and 3. [**Hint:** Notice that there are only 4 polynomials of degree 2 and 8 polynomials of degree 3 in $\mathbb{Z}_2[x]$.]