

MATH 103B – Modern Algebra II

Homework 4

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1. Let R be a commutative ring and let I be a maximal ideal of R . Prove that I is a prime ideal of R directly, without using Theorem 6.3 from the lecture notes. [**Hint:** If $rs \in I$ with $r \notin I$, consider $J := I + (r)$. Use this to express $1 = a + br$ for some $a \in I$ and $b \in R$, so $s = as + brs$.]

2. (a) Is the ideal $(x^2 + 3)$ of $\mathbb{R}[x]$ maximal?
(b) Is the ideal $(x^2 + 3)$ of $\mathbb{Z}[x]$ maximal?

3. Let $\varphi: R \rightarrow S$ be a ring homomorphism. Prove or disprove: if I is a maximal ideal of S , then

$$\varphi^{-1}(I) = \{r \in R \mid \varphi(r) \in I\}$$

is a maximal ideal of R . You must provide a proof or counterexample.

4. List all ideals of the following rings. Which ideals are prime? Which are maximal?

(a) $\mathbb{Z}_{10}$.

(b) $\mathbb{Z}_{25}$.

(c) $\mathbb{Q}$.

5. Compute the following sums and products of polynomials over finite rings.

(a) $(x^2 + \bar{2}x + \bar{1}) + (\bar{2}x^2 + 1)$ in $\mathbb{Z}_3[x]$.

(b) $(x^2 + \bar{2}x + \bar{1})(\bar{2}x^2 + 1)$ in $\mathbb{Z}_3[x]$.

(c) $(\bar{3}x^3 + \bar{3}x^2 - x) + (\bar{6}x^2 - \bar{8}x + \bar{4})$ in $\mathbb{Z}_9[x]$.

(d) $(\bar{3}x^3 + \bar{3}x^2 - x)(\bar{6}x^2 - \bar{8}x + \bar{4})$ in $\mathbb{Z}_9[x]$.

6. Let R be a ring and let $f, g \in R[x]$. Is it always true that

$$\deg(fg) = \deg(f) + \deg(g)?$$

If yes, you must provide a proof. If no, you must provide a counterexample. [**Hint:** Some of the examples from the previous question may be helpful.]

7. Let $f(x) := \bar{5}x^3 + \bar{6}x^2 - \bar{3}x + \bar{4}$ and $g(x) := x - \bar{2}$ be polynomials in $\mathbb{Z}_7[x]$. Use the division theorem for polynomial rings to find $q(x)$ and $r(x)$ with $\deg(r(x)) < \deg(g(x))$ such that

$$f(x) = g(x)q(x) + r(x).$$