

MATH 103B – Modern Algebra II

Homework 3

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1. Define a map

$$\begin{aligned}\varphi: \mathbb{Z}_2[x] &\rightarrow \mathbb{Z}_2[x] \\ f(x) &\mapsto f(x)^2.\end{aligned}$$

- (a) Prove that φ is a ring homomorphism.
- (b) If we replace $\mathbb{Z}_2[x]$ by $\mathbb{Z}[x]$, is the resulting map still a ring homomorphism? In other words, is the map

$$\begin{aligned}\psi: \mathbb{Z}[x] &\rightarrow \mathbb{Z}[x] \\ f(x) &\mapsto f(x)^2\end{aligned}$$

a ring homomorphism? Why or why not?

2. Show that $\mathbb{R}$ is not isomorphic (as a ring) to $\mathbb{C}$. [**Hint:** Recall that $i^2 = -1$ and use Proposition 5.3 from the lecture notes.]
3. Let $\varphi: R \rightarrow S$ be a ring homomorphism. Prove the following.
 - (a) If $I \trianglelefteq R$, then $\varphi(I) \trianglelefteq \text{Im}(\varphi)$, where

$$\varphi(I) := \{\varphi(x) \mid x \in I\}.$$

- (b) If $J \trianglelefteq S$, then $\varphi^{-1}(J) \trianglelefteq R$, where

$$\varphi^{-1}(J) := \{r \in R \mid \varphi(r) \in J\}.$$

- (c) If φ is an isomorphism, then φ^{-1} (the inverse map) is also an isomorphism. [**Hint:** You only need to check that φ^{-1} is a ring homomorphism, since it is automatically bijective.]
- (d) If R is commutative, then $\text{Im}(\varphi)$ is commutative.
- (e) If R is a field, then φ is injective.

4. Define a map

$$\begin{aligned}\varphi: \mathbb{Q}[x] &\rightarrow \mathbb{Q}[\sqrt{2}] \\ f &\mapsto f(\sqrt{2}),\end{aligned}$$

where $\mathbb{Q}[\sqrt{2}] := \{a + b\sqrt{2} \mid a, b \in \mathbb{Q}\}$.

- (a) Show that φ is a ring homomorphism.
- (b) Using the first isomorphism theorem, or otherwise, show that

$$\frac{\mathbb{Q}[x]}{(x^2 - 2)} \cong \mathbb{Q}[\sqrt{2}].$$

You may use the following without proof:

- If f is a polynomial and $f(a) = 0$, then $f = (x - a)g$ for some polynomial g .
- If $f \in \mathbb{Q}[x]$ and $f(\sqrt{2}) = 0$, then also $f(-\sqrt{2}) = 0$.

5. Let R be a ring. Prove that the map

$$\begin{aligned}\varphi: \mathbb{Z} &\rightarrow R \\ n &\mapsto n \cdot 1_R\end{aligned}$$

is a ring homomorphism.