

MATH 103B – Modern Algebra II

Homework 2

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1. Let $\mathbb{F}_4 = \{0, 1, a, b\}$ with addition and multiplication given by the following tables.

$+$	0	1	a	b	$\times$	0	1	a	b
0	0	1	a	b	0	0	0	0	0
1	1	0	b	a	1	0	1	a	b
a	a	b	0	1	a	0	a	b	1
b	b	a	1	0	b	0	b	1	a

Show that $\mathbb{F}_4$ is a field. [**Hint:** Notice that an equivalent way to define a field is the following: $\mathbb{F}$ is a field if $(\mathbb{F}, +)$ and $(\mathbb{F} \setminus \{0\}, \times)$ are abelian groups, and the multiplication operation of $\mathbb{F}$ distributes over its addition operation. Recall that $\mathbb{F} \setminus \{0\}$ is simply the set $\mathbb{F}$ without the element 0.]

2. Let R be a ring, and let $I \trianglelefteq R$.
- (a) Show that if I contains a unit of R , then $I = R$.
 - (b) Using part (a), or otherwise, deduce that if $I \neq R$, then I is **not** a subring of R . [**Hint:** if I is a subring of R , then $1_R \in I$.]

3. Let R be a commutative ring and let $x \in R$. Show that

$$(x) = Rx := \{rx \mid r \in R\}.$$

In other words, show that:

- (a) Rx is an ideal of R which contains x .
 - (b) If I is another ideal of R which contains x , then $Rx \subseteq I$.
4. Define

$$I := \{f(x) \in \mathbb{Z}[x] \mid f(0) \text{ is even}\},$$

so I is the set of polynomials with integer coefficients that have even constant term.

- (a) Show that $I \trianglelefteq \mathbb{Z}[x]$.
- (b) Show that I is not a principal ideal of $\mathbb{Z}[x]$. [**Hint:** If I were principal, then

$$I = (f(x)) = f(x)\mathbb{Z}[x] = \{f(x)g(x) \mid g(x) \in \mathbb{Z}[x]\}$$

for some $f(x) \in \mathbb{Z}[x]$. In particular, noting that $2 \in I$ and $x \in I$, this means that $2 = f(x)g(x)$ and $x = f(x)h(x)$ for some $g(x), h(x) \in \mathbb{Z}[x]$.]

5. Let $I, J \trianglelefteq R$.

(a) Show that $I \cap J$ is an ideal of R .

(b) Show that the *sum* of I and J

$$I + J := \{x + y \mid x \in I, y \in J\}$$

is an ideal of R .

6. Define

$$I := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a, b, c, d \in 2\mathbb{Z} \right\}.$$

(a) Show that I is an ideal of $M_2(\mathbb{Z})$.

(b) Show that every element of $M_2(\mathbb{Z})/I$ is equal to an element of the form

$$\begin{pmatrix} \varepsilon_1 & \varepsilon_2 \\ \varepsilon_3 & \varepsilon_4 \end{pmatrix} + I,$$

where $\varepsilon_i \in \{0, 1\}$. [**Hint:** Every matrix $\begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix} \in M_2(\mathbb{Z})$ can be written as

$$\begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix} = \begin{pmatrix} 2n_1 + \varepsilon_1 & 2n_2 + \varepsilon_2 \\ 2n_3 + \varepsilon_3 & 2n_4 + \varepsilon_4 \end{pmatrix} = \begin{pmatrix} 2n_1 & 2n_2 \\ 2n_3 & 2n_4 \end{pmatrix} + \begin{pmatrix} \varepsilon_1 & \varepsilon_2 \\ \varepsilon_3 & \varepsilon_4 \end{pmatrix}$$

with $\varepsilon_i = 0$ if x_i is even, and $\varepsilon_i = 1$ if x_i is odd.]