1. For \( p \) a prime number, a number field \( K \) is monogenic at \( p \) if there exists some \( \theta \in \mathcal{O}_K \) such that the ring homomorphism \( \mathbb{Z}[x] \to \mathcal{O}_K/p\mathcal{O}_K \) taking \( x \) to the class of \( \theta \) is surjective. Prove that if there exists a single prime of \( \mathcal{O}_K \) above \( p \) (ramified or not), then \( K \) is monogenic at \( p \).

2. Produce (e.g., by looking in LMFDB) an example of a number field \( K \) for which

\[
\mathcal{O}_K/2\mathcal{O}_K \cong \mathbb{F}_2 \times \mathbb{F}_2 \times \mathbb{F}_2,
\]

then use this to show that \( K \) is not monogenic at 2.

3. Let \( K \) be a number field and let \( R \) be a subring of \( \mathcal{O}_K \) which spans \( K \) as a \( \mathbb{Q} \)-vector space (i.e., an order of \( K \)). Let \( p \) be a prime number such that for every prime \( \mathfrak{p} \) of \( \mathcal{O}_K \) above \( p \),

(i) the inertia degree is 1, and

(ii) there exist some \( \lambda \in R \) such that \( v_{\mathfrak{p}}(\lambda) = 1 \) and \( v_q(\lambda) = 0 \) for all primes \( q \neq \mathfrak{p} \) of \( \mathcal{O}_K \) above \( p \).

Prove that the index \( [\mathcal{O}_K : R] \) is not divisible by \( p \). (This generalizes the argument used for cyclotomic fields.)

4. Neukirch, exercise I.9.1: Let \( L/K \) be a Galois extension of number fields such that \( \text{Gal}(L/K) \) is not cyclic. Prove that there are only finitely many primes of \( K \) that remain inert in \( L \).

5. Neukirch, exercise I.9.3: Let \( L/K \) be a (not necessarily Galois) extension of prime degree \( p \) with solvable Galois group. Suppose that \( \mathfrak{p} \) is a prime ideal of \( K \) which does not ramify in \( L \). Prove that if there are at least two primes of \( L \) above \( \mathfrak{p} \) of inertia degree 1, then \( \mathfrak{p} \) splits completely in \( L \).

6. For each of the following statements, find an example of a prime \( p \) and two quadratic extensions \( K \) and \( L \) of \( \mathbb{Q} \) exhibiting this particular behavior. Your four examples should be distinct. (You may use SageMath to verify the asserted properties.)

(a) The prime \( p \) can be totally ramified in \( K \) and \( L \) without being totally ramified in \( KL \).

(b) The fields \( K \) and \( L \) can both contain unique primes over \( p \), while \( KL \) does not.

(c) The prime \( p \) can be (unramified and) inert in both \( K \) and \( L \) without being inert in \( KL \).
(d) There can be (unramified) primes over $p$ of inertia degree 1 in both $K$ and $L$, but not in $KL$.

7. Let $L/K$ be a finite separable extension of fields with Galois closure $M$ and Galois group $G$. Put $H := \text{Gal}(M/L)$. Prove that $\bigcap_{x \in G} x^{-1} H x = \{e\}$.

8. Let $L/K$ be an extension of number fields with Galois closure $M$ and Galois group $G$. Put $H := \text{Gal}(M/L)$.

(a) Let $\mathfrak{p}$ be a prime ideal of $K$. Let $\mathfrak{q}$ be a prime ideal of $M$ above $\mathfrak{p}$. Show that the action of $G$ on the prime ideals of $M$ above $\mathfrak{p}$ induces a bijection between the double coset space $H \backslash G / G_\mathfrak{q}$ and the set of primes of $L$ above $\mathfrak{p}$.

(b) Suppose that $\mathfrak{p}$ does not ramify in $M$. Show that the inertia degree of the prime of $L$ above $\mathfrak{p}$ corresponding to the double coset $H x G_\mathfrak{q}$ equals the index $[G_\mathfrak{q} : G_\mathfrak{q} \cap x^{-1} H x]$.

(c) Optional: extend (b) to the case where $\mathfrak{p}$ may ramify in $M$. 