Math 204A (Number Theory), UC San Diego, fall 2022
Problem Set 3 – due Thursday, October 20, 2022

Submit at most five of the listed problems.

1. Let $R$ be a Dedekind domain. Prove that the following statements are equivalent.
   (a) The ring $R$ is a principal ideal domain.
   (b) The ring $R$ is a unique factorization domain.
   (c) The class group of $R$ is trivial.
   (Hint: for (b) $\implies$ (c), it is enough to show that every prime ideal $p$ is principal. First show that $p$ contains an irreducible element $\alpha$, then show that $\alpha$ generates a nonzero prime ideal.)

2. Let $L/K$ be a nontrivial extension of number fields such that $[\mathcal{O}_L : \mathcal{O}_K]$ is finite. Prove that $K$ is a totally real field (i.e., all of its archimedean embeddings are real) and $L = \mathbb{Q}(\sqrt{\alpha})$ for some $\alpha \in K$ such that $\tau(\alpha) < 0$ for every embedding $\tau : K \to \mathbb{R}$. Such a field $L$ is called a CM field (for “complex multiplication”).

3. Let $K$ be a number field of degree $n$ with signature $(r_1, r_2)$. We derive Minkowski’s improved estimate for the minimum norm of an ideal in an ideal class of $K$.
   (a) Prove that for any $t > 0$, the region
   $$X = \{(z_\tau) \in K_\mathbb{R} : \sum_\tau |z_\tau| < t\}$$
   has volume $2^{r_1} \pi^{r_2} \frac{t^n}{n!}$. (Hint: it may be easiest to set this up as a multiple integral, using polar coordinates for each complex embedding.)
   (b) Show that any nonzero ideal $I$ of $\mathcal{O}_K$ contains a nonzero element $\alpha$ satisfying
   $$|\text{Norm}_{K/\mathbb{Q}}(\alpha)| \leq \frac{n!}{n^n} \left(\frac{4}{\pi}\right)^{r_2} \sqrt{|d_K|}[\mathcal{O}_K : I].$$
   (Hint: apply Minkowski’s lattice point theorem to the region $X$ from (a), then use the arithmetic-geometric mean inequality.)
   (c) Deduce (as in lecture) that every ideal class of $K$ is represented by some integral ideal of norm at most $\frac{n!}{n^n} \left(\frac{4}{\pi}\right)^{r_2} \sqrt{|d_K|}$.

4. Let $K$ be a number field for which $|d_K| = 1$. Prove that $K = \mathbb{Q}$. (Hint: show that $\frac{n!}{n^n} \left(\frac{4}{\pi}\right)^n < 1$ for $n > 1$, then apply the previous exercise.)

5. Use the improved Minkowski estimate to show that $\mathbb{Q}(\sqrt{11})$ has class number 1.
6. Let $K$ be the number field $\mathbb{Q}[x]/(x^3 - x^2 + x + 1)$.
   
   (a) Find the home page of $K$ in the LMFDB.
   
   (b) Report what the LMFDB says about the signature of $K$ and the structure of the group $\mathfrak{o}_K^\times$ (including generators).
   
   (c) Prove that this answer is correct.

7. Repeat part (a) and (b) of the previous exercise for the fields
   
   $\mathbb{Q}(2^{1/3})$, $\mathbb{Q}(\sqrt{3}, \sqrt{5})$, $\mathbb{Q}(\zeta_5)$.
   
   (Hint: it may help to filter by signature and/or Galois group.)

8. (a) Use SageMath to list all the imaginary quadratic fields with discriminant in $[0, 10^4]$ with class number 1. The fact that there are no others of any discriminant is a deep theorem (the resolution of the Gauss class number one problem by Baker–Heegner–Stark).
   
   (b) Compute the fraction of real quadratic fields with discriminant in $[-10^4, 0]$ with class number 1. Note that this is quite far from zero!
   
   (c) Make a plot comparing two sets of data: the class number of an imaginary quadratic field $\mathbb{Q}(\sqrt{-D})$ as a function of $D$, and the class number times the logarithm of a fundamental unit of a real quadratic field $\mathbb{Q}(\sqrt{D})$ as a function of $D$. These should look much more similar.