Math 103A W23 HW 6

22. **Suggested answer**:

22. Let G be a group of order 20. If G has subgroups H and K of orders 4 and 5 respectively such that $hk = kh$ for all $h \in H$ and $k \in K$, prove that G is the internal direct product of H and K.

(i) $H \cap K = \{e\}$.

Suppose $x \in H \cap K$, then $x \in H$ and $x \in K$.

So $|x|$ divides $|H|$ and $|x|$ divides $|K|$

$\Rightarrow$ $|x|$ divides 4 and 5, then $|x| = 1$

$\Rightarrow$ $x = e$

So $H \cap K = \{e\}$.

(ii) $G = HK$. $|H| \cdot |K| = 4 \cdot 5 = 20$

we know that $H \times K$ = 20 elements, but we need to show they are distinct

$\Rightarrow$ If $h_1 k_1 \neq h_2 k_2$ then $h_1 \neq h_2$ or $k_1 \neq k_2$

$\Rightarrow$ Contrapositive: if $h_1 k_1 = h_2 k_2$ then $h_1 = h_2$ and $k_1 = k_2$

Given $h_1 k_1 = h_2 k_2 \Rightarrow h_2^{-1} h_1 = k_2 k_1^{-1}$

$h_2^{-1} h_1 \in H$ since H is a subgroup (inverse and closure properties)

Similarily, $k_2 k_1^{-1} \in K$.

$\Rightarrow h_2^{-1} h_1 = k_2 k_1^{-1} \Rightarrow H = K$ As proven in (i), $H \cap K = \{e\}$, so $H = K$ is only true when $H = K = e$.

$\Rightarrow h_2^{-1} h_1 = e$ and $k_2 k_1^{-1} = e$

$\Rightarrow h_1 = h_2$ and $k_2 = k_1$ as needed.

$\Rightarrow$ we have shown $G = HK = \{hk : h \in H, k \in K\}$

we are given that (iii) $hk = kh$ for all $h \in H$ and $k \in K$,
so G is the internal product of H and K.

Common mistakes:
Some people only showed $H \cap K = e$ and $|H||K| = 20 = |G|$. We also need to show these 20 hk pairs are distinct to claim that $|HK| = |G|$.

24. **Suggested answer**:

24.

Proposition 4. *There is not a non-cyclic abelian group of order* 51.

Proof. Any abelian group of order 51 is isomorphic to the direct product of cyclic groups whose orders are prime powers and multiply to 51.
The order of these cyclic groups must be 3 and 17, as the only other factors of 51 are 1 and 51 (and 51 is not a prime power).
Furthermore, we know that $\gcd(3, 17) = 1$ and therefore $\mathbb{Z}_3 \times \mathbb{Z}_{17}$ is cyclic.
In summary, any abelian group of order 51 is isomorphic to $\mathbb{Z}_3 \times \mathbb{Z}_{17}$ and thus cyclic. □

42. **Suggested answer**:

42.

Proposition. *$Inn(G)$ is a subgroup of $Aut(G)$.*

Proof. (Closure) Let i_g and i_h be inner automorphisms of G. Then for any $x \in G$ we have $(i_g \circ i_h)(x) = i_g(i_h(x)) = i_g(hxh^{-1}) = ghxh^{-1}g^{-1} = (gh)x(gh)^{-1}$. Thus, $\text{Inn}(G)$ is closed under the operation.

(Identity) The identity map of $\text{Aut}(G)$, i_e, is an inner automorphism because for any $x \in G$ we have $i_e(x) = exe^{-1} = x$. Thus, $\text{Inn}(G)$ contains an identity element.
(Inverse) Let i_g be an inner automorphism of G. Then for any $x \in G$ we have $(i_g^{-1} \circ i_g)(x) = i_g^{-1}(i_g(x)) = i_g^{-1}(gxg^{-1}) = g^{-1}(gxg^{-1})g = x$. Thus, $\text{Inn}(G)$ contains an inverse element.
Thus, $\text{Inn}(G)$ is a subgroup of $\text{Aut}(G)$. □

Comments:
In terms of correctness almost everyone did well, but a more structured answer (closure, identity and inverse sections clearly labeled) for such questions is always preferred!