Math 103A W23 HW 7

2. **Suggested answer**:

Q2. $f: C^* \to S$, $f(a+bi) = \begin{pmatrix} a & b \\ -b & a \end{pmatrix}$

1. We check if $f(xy) = f(x) \cdot f(y)$.

$f((a+bi)\cdot(c+di)) = f(ac-bd + (ad+bc)i)$

$= \begin{pmatrix} ac-bd & ad+bc \\ -(ad+bc) & ac-bd \end{pmatrix} = \begin{pmatrix} a & b \\ -b & a \end{pmatrix}\begin{pmatrix} c & d \\ -d & c \end{pmatrix} = f(a+bi)\cdot f(c+di)$ ✓

2. We check if there's bijectivity between them.

(a) One-to-one (Injective): Let $f(a+bi) = f(c+di) \to \begin{pmatrix} a & b \\ -b & a \end{pmatrix} = \begin{pmatrix} c & d \\ -d & c \end{pmatrix}$

$\to a=c,\ b=d$

$\to a+bi = c+di$ ✓

(b) Onto (surjective): For all $\begin{pmatrix} a & b \\ -b & a \end{pmatrix} \in S$, there exists $a, b \in \mathbb{R}$ where $a, b \neq 0$ since $\det(S) \neq 0$.

We can compute $f(a+bi)$ as $a+bi \in C^*$ to get $f(a+bi) = \begin{pmatrix} a & b \\ -b & a \end{pmatrix}$. Thus, it is onto.

Thus, C^* is isomorphic to S.

Comments:
Clearly structured answer. Note that the alternative of proving bijectivity is showing the well-defined inverse function of the isomorphism, which requires less hand-writing

11. **Suggested answer**:

Q11. $\mathbb{Z}_8$, $\mathbb{Z}_2 \times \mathbb{Z}_4$, $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$, D_4, Q_8

- $\mathbb{Z}_8$ is cyclic while $\mathbb{Z}_2 \times \mathbb{Z}_4$ and $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ are not.
- $\mathbb{Z}_2 \times \mathbb{Z}_4$ is not isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ because every element of $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ is of order 2 but not the case in $\mathbb{Z}_2 \times \mathbb{Z}_4$
- Any of the groups in $\{\mathbb{Z}_8, \mathbb{Z}_2 \times \mathbb{Z}_4, \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2\}$ can't be isomorphic to any group in $\{D_4, Q_8\}$ because $\{D_3, Q_8\}$ do not contain abelian groups while $\{\mathbb{Z}_8, \mathbb{Z}_2 \times \mathbb{Z}_4, \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2\}$ contains.
- Q_8 have six elements of order 4 while D_4 only has 2 two elements of order 4.

common mistakes:
Many people neglected to give justification for the differing group structures.

12. **Suggested answer**:

Q12. S_4 has elements of order either 1, 2, 3, 4, however D_{12} has elements of order 2, 3, 4, 6, 12. Therefore S_4 and D_{12} have different orders of their elements so they are not isomorphic.