Name: ____________________

Math 103B SP23 Midterm 1 Apr 26, 2023 **Student ID:** ______________

Unless otherwise specified, you have to provide justification to all of your answers.

1. Let $G = \mathbb{Z}/6$, $X = \{A, B, C, D, E\}$, and G acts on X by

$$1 \cdot A = B,\ 1 \cdot B = C,\ 1 \cdot C = A,\ 1 \cdot D = E,\ 1 \cdot E = D.$$

(a) (3 points) Find $2 \cdot B$ and $3 \cdot D$. Show your calculation.

Solution: $2 \cdot B = (1+1) \cdot B = 1 \cdot (1 \cdot B) = 1 \cdot C = A$.
$3 \cdot D = (2+1) \cdot D = 2 \cdot (1 \cdot D) = 2 \cdot E = 1 \cdot (1 \cdot E) = 1 \cdot D = E$.

(b) (2 points) Find $\mathcal{O}_E$, the orbit of E.

Solution: From (a), we can see that $\mathcal{O}_E = \{D, E\}$.

(c) (3 points) Find G_A, the stabilizer group of A.

Solution: From (a),

$$3 \cdot A = 2 \cdot B = A,$$

so $3 \in G_A$. A subgroup of $G = \mathbb{Z}/6$ containing 3 must be either $\{0, 3\}$ or the entire $\mathbb{Z}/6$. Since $1 \cdot A = B \neq A$, $1 \notin G_A$. Therefore, $G_A = \{0, 3\}$.

Remark: From (a), we can see that $\mathcal{O}_A = \{1, 2, 3\}$. Thus $|G_A| = 2$ by the orbit-stabilizer theorem. This can be used to assist the calculation of G_A as well.

2. (4 points) Solve the system of congruence

$$\begin{aligned} x &\equiv 5 \pmod{6} \\ x &\equiv 3 \pmod{7} \\ x &\equiv 2 \pmod{11}. \end{aligned}$$

Show your calculation. Your answer should be $x \equiv ?? \pmod{462}$ (here $462 = 6 \cdot 7 \cdot 11$).

Solution: We start with the largest modulo, i.e. 11. With $2 \pmod{11}$, we have 2, $2+11 = 13$, $13+11 = 24$. Now 24 is the first number in this list that is also 3 mod 7. We then add $7 \cdot 11 = 77$ to the numbers: 24, $24+77 = 101$. 101 happens to be also $5 \pmod{6}$, so the answer to the system is $x \equiv 101 \pmod{462}$.

3. Consider the homomorphism

$$\begin{aligned} \phi &: \mathbb{R}[x] \to \mathbb{R}^2 \\ \phi(f) &= (f(1), f(2)). \end{aligned}$$

(a) (2 points) Prove that ϕ is surjective.

Solution:

Proof. Let $(a, b) \in \mathbb{R}^2$. Then the line passing through $(1, a)$ and $(2, b)$ is a polynomial with $f(1) = a$ and $f(2) = b$. In particular, the line is given by $f(x) = (b - a)x + (2a - b)$. □

(b) (3 points) Using this homomorphism, prove that $\mathbb{R}[x]/(x^2 - 3x + 2)$ is isomorphic to $\mathbb{R}^2$.

Solution:

Proof. The kernel of this function is all polynomials $f(x)$ such that $f(1) = 0$ and $f(2) = 0$. From $f(1) = 0$, we know $x = 1$ is a root of $f(x)$, and thus $(x - 1)$ is a factor of $f(x)$. Similarly, $(x - 2)$ is a factor of $f(x)$. Thus $(x - 1)(x - 2)$ is a factor of $f(x)$, so $f(x) \in I$. On the other hand, if $f(x)$ is divisible by $x^2 - 3x + 2$, then $f(1) = f(2) = 0$. Therefore, the kernel of ϕ is I. By the first isomorphism theorem, the quotient $\mathbb{R}[x]/I$ is isomorphic to the image of ϕ. Since ϕ is surjective, we have that $\mathbb{R}[x]/I \simeq \mathbb{R}^2$. □

4. Let R be a commutative ring with 1.

(a) (3 points) Prove that if x is a nilpotent element, then $1 + x$ is a unit. Hint:

$$x^n - 1 = (x + 1)(x^{n-1} - x^{n-2} + \cdots + (-1)^{n-1}).$$

Solution:

Proof. Since x is nilpotent, there exists a positive integer n such that $x^n = 0$. Now,

$$(1 + x)(1 - x + x^2 - \cdots + (-1)^{n-1}x^{n-1}) = 1 - x^n = 1 - 0 = 1.$$

Thus $(1 + x)$ is a unit. □

(b) (3 points) Prove that if $e \in R$ is an idempotent, then $1 - e$ is also an idempotent.

Solution:

Proof. Let e be an idempotent. Thus $e^2 = e$ by definition. Thus $(1 - e)^2 = 1 - 2e + e^2 = 1 - 2e + e = 1 - e$, so $1 - e$ is also an idempotent. □

(c) (3 points) Consider $R = \mathbb{R}[x, y]$. Prove that (x) is a prime ideal but not a maximal ideal.

Solution:

Proof. The quotient $\mathbb{R}[x, y]/(x)$ is clearly isomorphic to $\mathbb{R}[y]$, which is an integral domain but not a field. Therefore, (x) is a prime ideal but not a maximal ideal. □

5. (a) (4 points) Prove that $\mathbb{Z}[\sqrt{-2}]$ is not isomorphic to $\mathbb{Z}[\sqrt{-3}]$.

Solution:

Proof. If $\phi : \mathbb{Z}[\sqrt{-2}] \to \mathbb{Z}[\sqrt{-3}]$ is an isomorphism, then $\phi(1) = 1$ since ϕ is surjective. Now $-2 = \phi(-2) = \phi(\sqrt{-2}\sqrt{-2}) = (\phi(\sqrt{-2})^2$, but there is no square root of -2 in $\mathbb{Z}[\sqrt{-3}]$. $\square$

(b) (2 points) Prove or disprove: $\mathbb{R}[\sqrt{-2}]$ is isomorphic to $\mathbb{R}[\sqrt{-3}]$.

Solution:

Proof. These are both equal to $\mathbb{C}$, so they are isomorphic.

Remark: for an explicit isomorphism, one can use $f(x+y\sqrt{-2}) = x+(y\sqrt{\frac{2}{3}})\sqrt{-3}$.

In terms of complex numbers, we simply have the identity map $f(z) = z$. $\square$