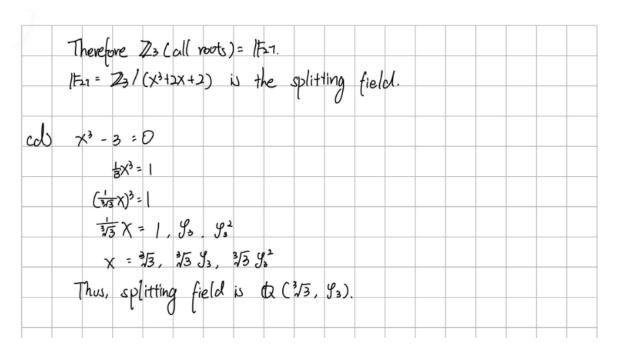
Ch 21.5
Q3.
$(a) x^4 - 10x^2 + 2 = 0$
$x^4 - 10x^2 + 25 = 2$
$(x^2 - \xi)^2 = 4$
$x^2-5=\pm 2$
$x^2 = 3, 7$
$x = \pm \sqrt{3}, \pm \sqrt{7}$
The splitting field is Q (13. 17)
cb) X ⁴ + 1 = 0
$X^{\mu} = -1$
X = 昼+昼i, 昼-昼i, -昼-唇i
The splitting field is Q(N), r)
(c) $(F_{27} = \mathbb{Z}_{2}/(X^{3}+2X+2) = \mathbb{Z}_{3}(X)$.
\overline{x} is a noot of x^3+2x+2 in $1/5$. Thus $\overline{x}^3+2\overline{x}+2=0$. chav $1/5$? = 3.
$(\overline{x}^3 + 2\overline{x} + 2)^5 = \overline{x}^9 + 6\overline{x}^7 + 6\overline{x}^6 + 12\overline{x}^5 + 24\overline{x}^4 + 20\overline{x}^3 + 24\overline{x}^2 + 24\overline{x} + 8$
$= \overline{x}^{9} + 20\overline{x}^{3} + 8$ $= \overline{x}^{9} + 2\overline{x}^{3} + 2$
$= (\overline{x}^3)^3 + 2\overline{x}^3 + 2$
$= (\frac{1}{1})^3 + \frac{1}{1}(\frac{1}{1})^3 + \frac{1}{1}(\frac{1})^3 + \frac{1}{1}(\frac{1}{1})^3 + \frac{1}{1}(\frac{1}{1})^3 + \frac{1}{1}(\frac{1}{1}$
=0
X+1 is a most of x+2x+2
Similarly, we get $(\frac{1}{x+1})^3 + 2(\frac{1}{x+1}) + 2 = 0$. Thus, $\frac{1}{x+1} = \frac{1}{x+2}$ is
a root of x3+2x+2.
Thus, Z3 Call roots) & F27. Since F27 = Z3(A), 157 & Z3 Call noots).



Q17

17.

Proposition 3. Let E be the algebraic closure of a field F. Then every polynomial p(x) in F[x] splits in E.

Proof. Suppose $p(x) \in F[x]$.

We will proceed with proof by induction on $\deg p(x)$.

Base case:

If deg p(x) = 1, then we have by definition that p(x) is itself a linear factor in E[x] (as $F \subset E$), and therefore p(x) splits in E.

Inductive hypothesis:

Suppose polynomials of degree n split in E.

Suppose $\deg p(x) = n + 1$.

We know that p(x) has some root α in an extension field E' of F.

Then we may observe that α is by definition algebraic over F and thus $\alpha \in E$ as E contains all the elements algebraic over F.

Hence p(x) has a root in E and thus $p(x) = (x - \alpha)g(x)$ with $g(x) \in E[x]$. Observe by the additivity of degree over multiplication that $\deg g(x) = n$. By the inductive hypothesis we have that g(x) splits in E, and thus we have shown that p(x) splits in E.

25.

Proposition 7. Let E be a field extension of F and $\alpha \in E$. Determine $[F(\alpha):F(\alpha^3)]$.

Proof. First observe that $f(x) = x^3 - \alpha^3 \in F(\alpha^3)[x]$ has a root at $x = \alpha$. As the minimal polynomial for α over $F(\alpha^3)$ by definition is the polynomial with α as a root of minimal degree, it follows that the minimal polynomial for α over $F(\alpha^3)$ has degree at most 3, and therefore

$$1 \le [F(\alpha) : F(\alpha^3)] \le 3$$

To show that this is indeed the tightest bound we can give, it suffices to provide examples of F and α with degree of extension 1, 2, and 3.

Example 1:

Let $F = \mathbb{Q}$ and $\alpha = 1$.

Obviously then $\alpha^3 = \alpha$ and thus $F(\alpha) = F(\alpha^3)$, or $[F(\alpha) : F(\alpha^3)] = 1$.

Example 2:

Let $F = \mathbb{Q}$ and $\alpha = \zeta_3$, the third root of unity.

We know $\Phi_3(x)$ is the minimal polynomial of ζ_3 over \mathbb{Q} and has degree 2. Furthermore $(\zeta_3)^3 = 1$, so $\mathbb{Q}((\zeta_3)^3) = \mathbb{Q}$.

Hence $[F(\alpha):F(\alpha^3)]=2$.

Example 3:

Let $F = \mathbb{Q}$ and $\alpha = \sqrt[3]{2}$.

Note Eisenstein's Criterion with p=2 gives us that $f(x)=x^3-2$ is irreducible, and thus f(x) is a degree 3 minimal polynomial of $\sqrt[3]{2}$ over \mathbb{Q} . Furthermore $(\sqrt[3]{2})^3=2$, so $\mathbb{Q}((\sqrt[3]{2})^3)=\mathbb{Q}$, and thus $[F(\alpha):F(\alpha^3)]=3$.

We have thus shown that the tightest bound is $1 \leq [F(\alpha) : F(\alpha^3)] \leq 3$. \square