Math 109: HW 5

For this homework, you can use the following facts. If it is a definition, you should mention “by definition” or “by definition of XXX (e.g. even)”. If it is not a definition, you do NOT have to cite.

1. Basic algebra, such as $2 + 2 = 4$, $1 - 3 = -2$, $2 \cdot 4 = 8$. This includes subtracting an integer, or dividing by a non-zero rational/real/complex numbers.

2. Common knowledge of whether numbers are integer/rational, e.g. $\frac{1}{2}$ is not an integer, $\pi$ is not rational, all integers are rational, all rational numbers are real, all real numbers are complex, etc.

3. **Associative law of addition**: $(a + b) + c = a + (b + c)$, and **associative law of multiplication**: $(ab)c = a(bc)$, for all $a, b, c \in \mathbb{C}$.

4. **Commutative law of addition**: $a + b = b + a$, and **commutative law of multiplication**: $ab = ba$, for all $a, b, c \in \mathbb{C}$.

5. **Distributive law**: For all $a, b, c \in \mathbb{C}$, we have $(a + b)(c + d) = ac + ad + bc + bd$. In particular, $(a + b)^2 = a^2 + 2ab + b^2$.

6. An integer $n$ is **even** if there exists an integer $k$ such that $n = 2k$.

7. An integer $n$ is **odd** if there exists an integer $k$ such that $n = 2k + 1$.

8. All integers are either even or odd.

9. A real number $x$ is **positive** if $x > 0$, and **negative** if $x < 0$.

10. All real numbers are either positive, negative, or 0.

11. Let $A, B$ be subsets of a universal set $U$.
   (a) We say $A$ is a **subset** of $B$, denoted by $A \subseteq B$, if for all $x \in A$, we have $x \in B$.
   (b) An element $x$ is in $A \cup B$, the **union** of $A$ and $B$, if $x \in A$ or $x \in B$.
   (c) An element $x$ is in $A \cap B$, the **intersection** of $A$ and $B$, if $x \in A$ and $x \in B$.
   (d) An element $x \in U$ is in $A^c$, the **complement** of $A$ in $U$, if $x \notin A$.

12. We denote by $\emptyset$ the **empty set**.

13. Let $A, B$ be sets. Then
   (a) $A \cap B \subseteq A$;
   (b) $A \subseteq A \cup B$. 

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14. A “function” \( f : A \to B \) is a **well-defined function**, or simply a function, if for all \( x \in A \), there is a unique \( y \in B \) such that \( f(x) = y \). (There are quotes here as \( f \) is not actually a function if it is not well-defined.)

15. A function \( f : A \to B \) is **injective** if for all \( x_1, x_2 \in A \), if \( f(x_1) = f(x_2) \), then \( x_1 = x_2 \).

16. A function \( f : A \to B \) is **surjective** if for all \( y \in B \), there exists \( x \in A \) such that \( f(x) = y \).

17. A function \( f : A \to B \) is **bijective** if it is injective and surjective.

Here are the questions.

1. Let \( A, B \) be subsets of a universal set \( U \). Prove that if \( A \subseteq B \), then \( B^c \subseteq A^c \). Be sure to consider the possibility that a set \( S \) could be empty before you write “Let \( x \in S \).”

2. Prove that if \( A, B \) are disjoint and \( C \subseteq B \), then \( A, C \) are disjoint. You may assume \( A, B, C \) are all non-empty.

3. Let \( A, B \) be set with \( A \) non-empty.
   (a) Prove that if \( A \subseteq B \), then \( A, B \) are not disjoint.
   (b) What if we remove the assumption “\( A \) is non-empty”? Is the statement in (a) still correct? Prove or provide a counterexample.

4. Let \( A, B, C \) be non-empty subsets of the universal set \( U \) such that \((A \cap B)^c \subseteq C \). Prove that \( A \subseteq B \cup C \).

5. For each of the following function, prove or disprove: is the function injective? Is it surjective?
   (a) \( f : \mathbb{Z} \to \{0, 1\} \)
      \[ f(n) = \begin{cases} 
      0 & \text{if } n \text{ is even,} \\
      1 & \text{if } n \text{ is odd.} 
      \end{cases} \]
   (b) \( g : \{0, 1\} \to \{1, -1\} \)
      \[
      \begin{align*}
      0 & \mapsto 1 \\
      1 & \mapsto -1
      \end{align*}
      \]
(c) 

\[ h : \{(x, y) \in \mathbb{R}^2 \mid xy = 1\} \to \mathbb{R} \]
\[ (x, y) \mapsto x \]

(d) 

\[ k : \{n \in \mathbb{Z} \mid n \geq 0\} \to \mathbb{Z} \]
\[ n \mapsto \begin{cases} \frac{n}{2} & \text{if } n \text{ is even}, \\ -\frac{n+1}{2} & \text{if } n \text{ is odd}. \end{cases} \]

6. For each of the following, is it a well-defined function? Explain.

(a) 

\[ \alpha : \mathbb{R} \to \mathbb{Z} \]
\[ \alpha(x) = x. \]

(b) 

\[ \beta : \mathbb{Z} \to \{0, 1, -1\} \]
\[ \beta(n) = \begin{cases} 1 & \text{if } n \text{ is even}, \\ -1 & \text{if } n \text{ is positive}, \\ 0 & \text{otherwise}. \end{cases} \]

(c) (Here \( \mathbb{R}_{\geq 0} \) is the set of non-negative real numbers.) 

\[ | - | : \mathbb{R} \to \mathbb{R}_{\geq 0} \]
\[ |x| = \begin{cases} x & \text{if } x \geq 0, \\ -x & \text{if } x \leq 0. \end{cases} \]

(d) 

\[ \gamma : \mathbb{R} \to \mathbb{R} \]
\[ \gamma(x) = \frac{x}{|x|}. \]

7. Let \( f : A \to B \) and \( g : B \to C \) be functions. For each of the following, prove or provide a counterexample.

(a) If \( g \circ f \) is injective, is \( f \) always injective?

(b) If \( g \circ f \) is injective, is \( g \) always injective?

(c) If \( g \circ f \) is surjective, is \( f \) always surjective?

(d) If \( g \circ f \) is surjective, is \( g \) always surjective?
8. Prove that for all positive integer \( n \), we have

\[
1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{n(n + 1)(2n + 1)}{6}.
\]

9. Define a sequence \( \{a_n\} \) by \( a_1 = 1 \), \( a_2 = 3 \), and \( a_{n+2} = a_{n+1} + a_n \). Prove that for all \( n \geq 1 \), we have

\[
a_n = \left( \frac{1 + \sqrt{5}}{2} \right)^n + \left( \frac{1 - \sqrt{5}}{2} \right)^n.
\]

**Appendix: Some \LaTeX code**

1. brace for set \( \{ \} \): \{ \}
2. element of a set \( \in, \notin \): \in, \not\in
3. set union \( \cup \) and set intersection \( \cap \): \cup and \cap
4. subset \( \subseteq \): \subseteq
5. empty set \( \emptyset \): \emptyset
6. function composition \( g \circ f \): \circ f
7. binomial coefficients \( \binom{n}{r} \): \binom{n}{r}