Math 109: HW 4

For this homework, you can use the following facts **without** citing.

1. Basic algebra, such as $2 + 2 = 4$, $1 - 3 = -2$, $2 \cdot 4 = 8$, $0 \cdot x = 0$, $1 \cdot x = x$, $2 \cdot x = x + x$.

2. For all complex numbers $x, y, z$, we have the **Associative Law**: $(x + y) + z = x + (y + z)$ and $(xy)z = x(yz)$; thus we are allowed to write $x + y + z$ and $xyz$. We also have the **Commutative Law**: $x + y = y + x$ and $xy = yx$.

3. **Distributive law**: For all $a, b, c \in \mathbb{C}$, we have $(a + b)(c + d) = ac + ad + bc + bd$. In particular, $(a + b)^2 = a^2 + 2ab + b^2$.

4. Let $a, b, c$ be real numbers with $b > c$. Then $a + b > a + c$.

5. Let $a, b, c$ be real numbers with $b > c$. If $a > 0$, then $ab > ac$. If $a < 0$, then $ab < ac$. Same applies if we use non-strict inequalities $\geq$.

6. Let $a, b, c$ be complex numbers. If $ab = 0$, then $a = 0$ or $b = 0$. If $a^2 = 0$, then $a = 0$. If $ab = ac$, then $a = 0$ or $b = c$.

7. Let $a$ be a positive real number and $n$ be a positive integer. We define $a^{\frac{1}{n}}$ to be the positive real number that is the solution to the equation $x^n = a$. (It can be shown that the solution exists and is unique, hence it is ok to say “the” here).

Now here are some definitions / facts that you **need** to cite when used. The citing instructions are given after the list.

1. 1 is the smallest positive integer. That is, there does not exist an integer $n$ such that $n > 0$ and $n < 1$.

2. An integer $n$ is **even** if there exists an integer $k$ such that $n = 2k$.

3. An integer $n$ is **odd** if there exists an integer $k$ such that $n = 2k + 1$.

4. All integers are even or odd.

5. A real number $x$ is **positive** if $x > 0$, and **negative** if $x < 0$.

6. All real numbers are either positive, negative, or 0.

7. A real number $x$ is **rational** if there exists integers $m, n$ with $n \neq 0$ such that $x = \frac{m}{n}$.

When you use any of these, cite “HW4 Facts” and the numbering, or the name of what you are citing (the bold words). You can also use the statement of the previous questions to solve later ones, by citing “HW4 Q (problem number)”. However, you are **not** allowed to use other facts you know.

1. Prove that 1 is not even. (Recall from class that we have proved the fact that if $n > 0$, then $2n > n$. You can use this fact. )
2. Prove that if \( n \) is an odd integer, then \( n \) is not even. Then explain why “if \( n \) is an even integer, then \( n \) is not odd” is true.

3. Let \( n \) be an integer. Prove that if \( 5n \) is odd, then \( n \) is odd.

4. Prove that if \( a, b \) are positive real numbers with \( a \neq b \), then

\[
\frac{1}{a} + \frac{1}{b} \neq \frac{4}{a + b}.
\]

(Since we are talking about real numbers, it is ok to subtract and divide, as long as we are not dividing by 0.)

5. Prove that \( \sqrt{2} \) is irrational. You can use the fact that \( \sqrt{2} \) is irrational.

6. We define the smallest positive real number to be a positive real number \( x \) such that for all positive real number \( y \), we have \( x \leq y \). Prove that the smallest positive real number does not exist.

7. Let \( A, B, C \) be sets.
   1. Prove that \( A \subseteq A \cup B \).
   2. Prove that \( A \cap B \subseteq A \).
   3. Prove that if \( A \subseteq B \) and \( B \subseteq C \), then \( A \subseteq C \).

8. Let \( A, B \) be subsets of a universal set \( U \). Prove that if \( A \cap B^c = \emptyset \), then \( A \subseteq B \).

Appendix: Some \LaTeX\ Commands

1. \( \geq \) and \( \leq \): \texttt{\geq} or \texttt{\ge}; and \texttt{\leq} or \texttt{\le}
2. brace \{\}: \texttt{\{\}}
3. square root, \( n \)-th root \( \sqrt{2} \), \( \sqrt[2]{2} \): \texttt{\sqrt{2}}, \texttt{\sqrt[n]{2}}
4. set union \( \cup \) and set intersection \( \cap \): \texttt{\cup} and \texttt{\cap}
5. subset \( \subseteq \): \texttt{\subseteq}
6. not a subset \( \nsubseteq \): \texttt{\not\subseteq}
7. empty set \( \emptyset \): \texttt{\emptyset}
8. blackboard bold \( \mathbb{Z}, \mathbb{R} \): \texttt{\mathbb{Z}}, \texttt{\mathbb{R}}