Math 109: HW 2

Re-read the instructions for proof from HW 1.

For this homework, you are allowed to use the following definitions / facts. First, we have facts that you are free to use without citing.

1. Suppose $x, y, z$ are integers. Equality is reflexive: $x = x$; symmetric: if $x = y$ then $y = x$; and transitive: if $x = y$ and $y = z$, then $x = z$.

2. (The set of integers is closed under addition) If $a, b$ are integers, there is an integer $c$ such that $c = a + b$. (This is an indirect way to say that $a + b$ is an integer.) Similarly, (the set of integers is closed under multiplication) there is an integer $d$ such that $d = ab$.

3. Basic algebra of explicit numbers is allowed. That is, you can say $1 + 1 = 2$, $2 + 2 = 4$, $2 \cdot 2 = 4$, etc. You need to be more careful when the numbers are not explicit (i.e. with $a$, $n$, etc.).

4. For all integers $n$, we have $n + n = 2n$, where the right hand side is viewed in terms of multiplication $2 \cdot n$.

Now here are some definitions / facts that you need to cite when used. The citing instructions are given after the list.

1. For all integers $x, y, z$, we have the **Associative Law of Addition**: $(x + y) + z = x + (y + z)$; thus we are allowed to write $x + y + z$. We also have the **Commutative Law of Addition**: $x + y = y + x$.

2. (Additive Inverse) For all integers $x$, there exists an integer $x'$ such that $x + x' = 0$. (Note: we did not state that this integer is unique. Nor did we state that this integer is the same as $(-1) \cdot x$. Nor did we say that there is an integer $x'$ such that $x' + x = 0$.)

3. For all integers $x, y, z$, we have the **Associative Law of Multiplication**: $(xy)z = x(yz)$; thus we are allowed to write $xyz$. We also have the **Commutative Law of Multiplication**: $xy = yx$.

4. (Distributive Law) For all integers $a, b, c$, we have $(a + b)c = ac + bc$.

5. An integer $n$ is **even** if there exists an integer $k$ such that $n = 2k$.

6. An integer $n$ is **odd** if there exists an integer $k$ such that $n = 2k + 1$.

7. An integer $n$ is **positive** is $n > 0$, and **negative** if $n < 0$.

8. If $n$ is a negative integer, there exists a positive integer $k$ such that $n = -k$. Here $-k$ means $(-1) \cdot k$.

9. (Trichotomy) All integers are either positive or negative or 0.
10. (Transitivity of inequalities) For all integers \(x, y, z\) with \(x < y\) and \(y < z\), we have \(x < z\).

11. For all integers \(x, y, z\) such that \(x > 0\) and \(y > z\), we have \(xy > xz\).

12. For all integers \(x, y\) with \(x > y\), we have \(-x < -y\). Here \(-x\) and \(-y\) mean \((-1) \cdot x\) and \((-1) \cdot y\) respectively.

When you use any of these, cite “HW2 Facts” and the numbering, or the name of what you are citing (the bold words). You can also use the statement of the previous questions to solve later ones, by citing “HW2 Q (problem number)”. For instance, you’ll find Q2 useful for Q4. However, you are not allowed to use other facts you know.

Here is an example.

**Proposition 1.** If \(n\) is an integer, then \(2n\) is an even integer.

**Proof.** Let \(n\) be an integer. Then there exists an integer \(k\), namely \(k = n\), such that \(2n = 2k\). By HW2 Fact 6, we have shown that \(2n\) is an even integer. \(\Box\)

1. Write the very first sentence of a proof for the following statements, to address the “if …”. You do not have to unwind anything in the if-statement, i.e. you do not have to address “positive” or “odd”.

   (a) If \(x\) is a positive integer, then \(x + 1\) is a positive integer.

   (b) If \(x, y\) are odd integers, then \(x + y\) is an even integer.

2. Use a know-show table to prove that: for all integers \(a, b, c\), we have that \(a(b+c) = ab+ac\).

3. Use a know-show table to prove that: if \(m\) is an even integer and \(n\) is an odd integer, then \(m + n\) is an odd integer.

4. Prove that for all integers \(a, b, c, d\), we have that \((a + b)(c + d) = ac + ad + bc + bd\). As a corollary, prove that if \(a, b\) are integers, then \((a + b)^2 = a^2 + 2ab + b^2\).

5. Prove that if \(a\) is an odd integer, then \(a^2\) is an odd integer.

6. Prove that if \(a, b\) are integers with \(a + b = a\), then \(b = 0\). (Be careful at citing!)

7. Prove that for all positive integers \(a, b\), we have that \(ab\) is positive.

8. Prove that if \(a\) is a positive integer and \(b\) is a negative integer, then \(ab\) is negative.

9. Prove that if \(a\) is a negative integer and \(b, c\) are integers with \(b > c\), then \(ab < ac\). (Be careful at citing! Also, note that we do not have any fact about addition and inequality together. Do not use any of those.)