Groups 32

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Sample Problems with Solutions

The purpose of this handout is to show how Groups32 can help a student solve problems. All problems are solved using a combination of work with the Groups32 and theory. Since these problems may be tackled by students in different courses and at different times within the the course, don’t be surprised if you do not yet know some of the theorems or concepts mentioned.

In Mathematics there are usually two problems:

1. Knowing what is true
2. Proving it is true

I find experimentation useful to know what is true.

I find it easier to prove what is true once I know what is true.

PROBLEM 1

Show that a group of order 9 is isomorphic to either Z_9 or $Z_3 \times Z_3$.

SOLUTION

According to the CHART command, there are only two groups of order 9: They are the groups numbered 15 and 16.

```
G1>> CHART
Order                 Groups of that order
  1   1                 * = non-abelian
  2   2
  3   3
  4   4   5
  5   6
  6   7   8*
  7   9
  8  10  11  12  13* 14*
  9  15  16
```

So the problem becomes, how do we show that one of these is Z_9 while the other is Z_3 x Z_3?

We take a closer look at these two groups using the ORDERS command:

```
G1>> ORDERS   for Group Number 15
Group number 15 of Order 9
    1 elements of order  1:   A
    2 elements of order  3:   D G
    6 elements of order  9:   B C E F H I

G15>> ORDERS   for Group Number 16
Group number 16 of Order 9
    1 elements of order  1:   A
    8 elements of order  3:   B C D E F G H I
    0 elements of order  9:
```

Notice that Group 15 has several elements of order 9 -- for example B. This means that B, B^2, ... , B^8 and B^9 = A are all different -- and therefore are all the elements of the group. B is a generator. The map $k \to B^k$ gives an isomorphism from Z_9 to Group 15.

In general: If a group of order n has an element of order n, then the group is isomorphic to Zn. Conversely any group isomorphic to Zn has an element of order n.

Other ways to see that B generates Group 15:

1. You can use the POWERS command to see that the powers of B are all the elements of the group.
2. You can use the GENERATES command to show that the subgroup generated by B is the whole group.
3. You can use the SUBGROUPS command to see that the whole group is generated by a single element.)

So, how do we see that Group 16 is isomorphic to Z3 x Z3?

First way: (Circumstantial Evidence)

1. Z_3 x Z_3 is not isomorphic to Z_9 (because it doesn't have any element of order 9)
2. You have been told that every group of order 1-32 must be isomorphic to one of the groups in Groups32 -- and math instructors always speak the truth.
3. Z_3 x Z_3 is not isomorphic to group 15 -- so it therefore must be isomorphic to group 16

Second way: (comparing tables)

1. Figure out some way to get a multiplication table for Z_3 x Z_3
2. Use the ISOMORPHISM command to explicitly find an isomorphism

Here is one way to get a table for Z_3 x Z_3: The PERMGROUPS subpackage can produce tables for groups defined by permutations. So here is a trick for dealing with products of groups: If you know how to get G_1 by permutations, and you know how to get G_2 by permutations -- can you think of a way to get G_1 x G_2?

You might want to think about this before looking at the script. I will just give the solution without any comments (so you will have to think about why it works and how to do this in general).

```
G16>> PERMGRPS

  CREATE    ELEMENTS  HELP      INFO
  INSTALL   MAIN      MULTIPLY  QUIT
  X

G16>> CREATE
Subgroup of Sn -- what is n?  Number 6
     Put in generators as product of cycles.
     End with a blank line
Generator (1 2 3)
Generator (4 5 6)
Generator
Group is of order 9
A  ()                 B  (4 5 6 )               C  (4 6 5 )
D  (1 2 3 )           E  (1 2 3 )(4 5 6 )       F  (1 2 3 )(4 6 5 )
G  (1 3 2 )           H  (1 3 2 )(4 5 6 )       I  (1 3 2 )(4 6 5 )

G16>> INSTALL
    Install as table k (1..5)  Number 1
G1>> MAIN
```

Now that a table for Z_3 x Z_3 is installed as group 1, you can use the ISOMORPHISM command to find an explicit isomorphism. [Because of the erasing of the screen, I will not be showing the steps in constructing isomorphisms.] You will find that sending B to B and D to D works. This is not the only possibility.

Third way: (Examining Subgroups)

There is a theorem which says that if a group G has two subgroups H and K satisfying the following conditions:

(i) H and K are normal subgroups
(ii) $H \cap K = (e)$
(iii) G = HK

Then G is isomorphic to H x K.

Use the SUBGROUPS command to list the subgroups of group 16:

```
G1>> SUBGROUPS   of Group Number 16
  ... wait

   * = Normal subgroup
   Generators                     Subgroup
  0  { }                          *{ A }
  1  { B }                        *{ A B C }
  2  { D }                        *{ A D G }
  3  { F }                        *{ A F H }
  4  { E }                        *{ A E I }
  5  { B D }                      *{ A B C D E F G H I }
```

Notice that H = subgroup 1 and K = subgroup 2 have all the required properties.

(Theorems can be marvelous labor-saving devices. In that regard they have much in common with computers.)

Fourth way: (independent of Groups32)

If o(G)=9 and if G has an element of order 9, then G is isomorphic to Z_9.

Suppose that G does not have any element of order 9. It has, therefore, the identity of order 1 and every other element has order 3 (why?). So we must try to show that any such group is isomorphic to Z_3 x Z_3.

There are a number of ways to proceed. It might be helpful, for example, to show that the group G must be abelian. (Actually all groups of order p^2 are abelian, when p is a prime). If you do this then let b be an element of order 3 and c be an element not in the subgroup generated by b. You should be able to show that $(i,j) \rightarrow b^i c^j$ is an isomorphism from Z_3 x Z_3 to G.

PROBLEM 2

For which n is there only one group of order n?

What about the conjecture that this is true if and only if n is prime?

I used the CHART command and made a list of the groups with this property.

```
 1   1          * = non-abelian
 2   2
 3   3
 5   6
 7   9
11  19
13  25
15  28
17  43
19  49
23  59
29  88
31  93
```

Well, 2 is a prime, 3 is a prime, 5 is a prime, 7 is a prime, 11 is a prime, 13 is a prime, 15 is not a prime, 17 is a prime, ... The conjecture seems to be true up to a certain amount of experimental error.

This comment is inspired by a famous joke about a mathematician, physicist, and engineer.

It is true that there is only one group of order p when p is a prime. You should be able to prove this. We have just shown that the converse is not true.

Can anything be said about those n for which there is only one group of order n?

PROBLEM 3

There are some n for which the groups occur in a pair -- one abelian and one non-abelian. Is there anything that can be said about such n?

Again the CHART command was used and a list is made:

```
 6   7   8*
10  17  18*
14  26  27*
21  55  56*
22  57  58*
26  77  78*
```

For even n we see (perhaps) a pattern -- it seems that $n = 2p$ for p a prime. The abelian group is Z_n (or course). Check that the non-abelian group turns out to be D_p (the dihedral group). Can you prove that these are the only two possible groups of order 2p when p is a prime?

PROBLEM 4

One characterization of a normal subgroup is that every right coset is a left coset. Just to see what it looks like, find a non-normal subgroup of a group -- and a right coset which is not equal to any of the left cosets.

We need to look at a non-abelian group -- so the first possibility is group 8 (which is of order 6 and is isomorphic to S_3).

```
G14>> SUBGROUPS   of Group Number 8
  ... wait

   * = Normal subgroup
   Generators             Subgroup
  0  { }                  *{ A }
  1  { D }                 { A D }
  2  { E }                 { A E }
  3  { F }                 { A F }
  4  { B }                *{ A B C }
  5  { B D }              *{ A B C D E F }
```

```
G8>> COSETS   of subg generated by set: { d }

 Left Cosets                 Right Cosets
{ A D }                     { A D }
{ B F }                     { B E }
{ C E }                     { C F }

The subgroup { A D }  is NOT a NORMAL subgroup
```

The right coset containing B and E does not coincide with any of the left cosets.

PROBLEM 5

If a group is abelian, then every subgroup is a normal subgroup. A textbook raises the question of whether the converse is true: If every subgroup in a group is normal, must the group be abelian?

There are two possibilities: either the converse is true or it isn’t. If it is not true it should be possible to search for a counter-example. Of course, if the converse is actually true, you will never find one -- and will grow very old looking (but at least you will be having fun).

As it turns out, you do not have to look very far. The first example of a non-abelian group all of whose subgroups are normal is group 14 (of order 8).

```
G13>> SUBGROUPS   of Group Number 14
  ... wait

   * = Normal subgroup
   Generators               Subgroup
  0  { }                    *{ A }
  1  { C }                  *{ A C }
  2  { B }                  *{ A B C D }
  3  { E }                  *{ A C E G }
  4  { F }                  *{ A C F H }
  5  { B E }                *{ A B C D E F G H }
```

Such groups are called Hamiltonian.

The two non-abelian groups of order 8 probably come up one way or another in your text. One of them is the dihedral group D_4. The other is called the Quaternionic Group (or the group of Quaternionic Units). Which one is this?

PROBLEM 6

Show that the groups D_6 and D_3 x Z_2 are isomorphic.

We can construct both groups as permutation groups and then use ISOMORPHISM to show that they are isomorphic.

```
G8>> PERMGRPS

  CREATE    ELEMENTS  HELP      INFO
  INSTALL   MAIN      MULTIPLY  QUIT
  X

G8>> CREATE
Subgroup of Sn -- what is n?  Number 5
     Put in generators as product of cycles.
     End with a blank line
Generator (1 2 3)
Generator (1 2)
Generator (4 5)
Generator
Group is of order 12
A  ()                     B  (4 5 )                C  (2 3 )
D  (2 3 )(4 5 )           E  (1 2 )                F  (1 2 )(4 5 )
G  (1 2 3 )               H  (1 2 3 )(4 5 )        I  (1 3 2 )
J  (1 3 2 )(4 5 )         K  (1 3 )                L  (1 3 )(4 5 )

G8>> INSTALL
    Install as table k (1..5)  Number 1
```

At this point we have D_3 x Z_2 installed as group 1.

```
G1>> CREATE
Subgroup of Sn -- what is n?  Number 6
     Put in generators as product of cycles.
     End with a blank line
Generator (1 2 3 4 5 6)
Generator (2 6)(3 5)
Generator
Group is of order 12
A  ()                        B  (2 6 )(3 5 )           C  (1 2 )(3 6 )(4 5 )
D  (1 2 3 4 5 6 )            E  (1 3 )(4 6 )           F  (1 3 5 )(2 4 6 )
G  (1 4 )(2 3 )(5 6 )        H  (1 4 )(2 5 )(3 6 )     I  (1 5 )(2 4 )
J  (1 5 3 )(2 6 4 )          K  (1 6 5 4 3 2 )         L  (1 6 )(2 5 )(3 4 )

G1>> INSTALL
    Install as table k (1..5)  Number 2
```

At this point we have D_6 installed <u>as group 2</u>

```
G2>> MAIN

  CENTER    CENTRALI  CHART     CONJ-CLS
  COSETS    EVALUATE  EXAMPLES  GENERATE
  GROUP     HELP      INFO      ISOMORPH
  LEFT      NORMALIZ  ORDERS    PERMGRPS
  POWERS    QUIT      RIGHT     STOP
  SUBGROUP  TABLE     X

G2>> ORDERS   for Group Number 1

Group number 1 of Order 12
    1 elements of order  1:   A
    7 elements of order  2:   B C D E F K L
    2 elements of order  3:   G I
    0 elements of order  4:
    2 elements of order  6:   H J
    0 elements of order 12:

G1>> ORDERS   for Group Number 2

Group number 2 of Order 12
    1 elements of order  1:   A
    7 elements of order  2:   B C E G H I L
    2 elements of order  3:   F J
    0 elements of order  4:
    2 elements of order  6:   D K
    0 elements of order 12:
```

Using the orders of the elements as a guide, we obtain a isomorphism between these groups by sending $H \to D$ and $C \to C$.

REMARK: We have here a correct proof. We have explicitly produced an isomorphism. This is not a particularly insightful proof - so now that we know it is true, it might be important to find a proof that uses the properties of these groups (perhaps geometric reasoning). What we hope to gain is an idea of whether this might be a special case of a more general theorem -- and a way of proving the more general theorem (if there is one). You might look at the dihedral group as the symmetries of a regular polygon to see if you can prove this theorem and find a generalization.

CLOSING COMMENTS

1. Don’t be discouraged if you do not yet know some of the things discussed in this handout -- or if you find that you don’t yet have the tools to prove some of your conjectures. Experiments can serve to provide you with things to look for when they come up in class -- and, in fact, Groups32 was intended as much to raise questions as to provide tools for answering them.
2. Even in the most theoretical parts of mathematics, both theorems and proofs are often suggested by the intuition that comes from experience with examples. Be aware, however, that 32 is a very small number -- so don’t run amuck making conjectures based upon this relatively small set of groups. (For example, there are indeed *simple* groups which are not cyclic of prime order -- but the first example is A_5 which is of order 60).

   Nevertheless, I was amazed to find that even an earlier version of this program, which had groups only up to order 16, was still quite useful.

3. **A SUGGESTION**
   In Groups32 all groups are identified by a number. In most texts (and in other parts of the real world) groups have common names. Keep a list of numbers 1-144 and, as you identify them, the common name (or names) of each group you identify. If you manage to identify a lot of them -- please send me the list!