

Introduction to Groups32

A group theory calculator

Prof John J Wavrik
Dept of Mathematics
Univ of Calif – San Diego

Groups32 contains the tables for the groups of orders 1-32 (there are 144 of them).

	<u>A</u>	<u>B</u>	<u>C</u>
A	A	B	C
B	B	C	A
C	C	A	B

It uses the tables to calculate information about these groups. It uses essentially the same methods that you would use by hand - but a computer is much faster than a human at doing this type of computation.

An Early Theorem

Theorem:

In the table for a group, every element appears exactly once in each row and in each column.

Prove this theorem

I was once asked if this condition is enough:

Suppose that we have a table in which one element is the identity and in which every element appears exactly once in each row and each column. Must this be the table of a group?

The only thing that can fail is associativity.

The answer is NO - and here is an example:

	<u>A</u>	<u>B</u>	<u>C</u>	<u>D</u>	<u>E</u>
<u>A</u>	A	B	C	D	E
<u>B</u>	B	A	D	E	C
<u>C</u>	C	D	E	B	A
<u>D</u>	D	E	A	C	B
<u>E</u>	E	C	B	A	D

We check that this operation is not associative

	<u>A</u>	<u>B</u>	<u>C</u>	<u>D</u>	<u>E</u>
A	A	B	C	D	E
B	B	A	D	E	C
C	C	D	E	B	A
D	D	E	A	C	B
E	E	C	B	A	D

(BB)C

B(BC)

()C

B()



	<u>A</u>	<u>B</u>	<u>C</u>	<u>D</u>	<u>E</u>
A	A	B	C	D	E
B	B	A	D	E	C
C	C	D	E	B	A
D	D	E	A	C	B
E	E	C	B	A	D

(BB)C

B(BC)

(**A**)C

B()



	<u>A</u>	<u>B</u>	<u>C</u>	<u>D</u>	<u>E</u>
A	A	B	C	D	E
B	B	A	D	E	C
C	C	D	E	B	A
D	D	E	A	C	B
E	E	C	B	A	D

(BB)C

B(BC)

(**A**)C

B(**D**)



	<u>A</u>	<u>B</u>	<u>C</u>	<u>D</u>	<u>E</u>
A	A	B	C	D	E
B	B	A	D	E	C
C	C	D	E	B	A
D	D	E	A	C	B
E	E	C	B	A	D

(BB)C

B(BC)

(A)C

B(D)

C

	<u>A</u>	<u>B</u>	<u>C</u>	<u>D</u>	<u>E</u>
A	A	B	C	D	E
B	B	A	D	E	C
C	C	D	E	B	A
D	D	E	A	C	B
E	E	C	B	A	D

(BB)C

B(BC)

(A)C

B(D)

C

E

Groups32 uses an internal set of tables for the groups of orders 1-32 and offers a collection of commands

Here are the Commands

CENTER	CENTRALIZER	CHART	CONJ-CLS
COSETS	EVALUATE	EXAMPLES	GENERATE
GROUP	HELP	IDENTIFY	INFO
ISOMORPHISM	LEFT	NORMALIZER	NUMTHEORY
ORDERS	PERMGRPS	POWERS	QUIT
RIGHT	SEARCH	STOP	SUBGROUPS
TABLE	X		

To apply a command, you start typing letters. Once you have typed enough letters to distinguish the command from others, Groups32 will automatically complete the command and ask you to put in numbers or letters as needed.

Example:

The **CHART** command will list groups of various orders. There are other commands that begin with the letter **"C"** but this is the only one which begins with **"CH"**. Once you type **"C"** and **"H"** the command is completed for you.

G1>> CH

You type in the letters C and H
(your input is shown in red)

G1>> CHART Order of Groups (1-32 or 0) Number

Groups32 completes the CHART command for you.

It then informs you that you can put in a number from 1-32 if you want to see a list of groups of a specific order, or put in the number 0 if you want a listing of all groups.

```
G1>> CHART    Order of Groups (1-32 or 0) Number 8
      10     11     12     13*   14*
      There are 5 Groups of order 8
      3 abelian and 2 non-abelian
```

If you put in the number 8, the program shows you that there are a total of 5 groups of order 8. The groups numbered 10, 11 and 12 are abelian (commutative) while those numbered 13 and 14 are non-abelian.

Helpful Hints

- Type `H` (for Help) at any time and you will get the list of commands.
- Type `INFO` or `X` and then a command and you will get information about that command (the command will not execute).

G1>> **HELP**

CENTER	CENTRALIZER	CHART	CONJ-CLS
COSETS	EVALUATE	EXAMPLES	GENERATE
GROUP	HELP	IDENTIFY	INFO
ISOMORPHISM	LEFT	NORMALIZER	NUMTHEORY
ORDERS	PERMGRPS	POWERS	QUIT
RIGHT	SEARCH	STOP	SUBGROUPS
TABLE	X		

G1>> X

This will provide information about the next command you use. INFO and X do the same thing but X is quicker to type.

G1>> CHART

This prints a chart of all groups of a given order.

The groups are numbered 1 to 144 and are grouped by order. An asterisk indicates that the group is not abelian.

Input of 0 for the order gives all groups.

A Short Sample Session

A separate "Sample Session" file goes through many of the commands in Groups32. You should go through this file while running Groups32 and try things for yourself.

However - here is a quick sample

```
G1>> CHART    Order of Groups (1-32 or 0) Number 4
      4      5
      There are 2 Groups of order 4
      2 abelian and 0 non-abelian
```

There are two essentially different tables for groups of order 4. Both represent abelian groups. The groups are number 4 and number 5.

G1>> TABLE for Group Number 4

	<u>A</u>	<u>B</u>	<u>C</u>	<u>D</u>
A	A	B	C	D
B	B	C	D	A
C	C	D	A	B
D	D	A	B	C

G4>> TABLE for Group Number 5

	<u>A</u>	<u>B</u>	<u>C</u>	<u>D</u>
A	A	B	C	D
B	B	A	D	C
C	C	D	A	B
D	D	C	B	A

Table 4

	<u>A</u>	<u>B</u>	<u>C</u>	<u>D</u>
A	A	B	C	D
B	B	C	D	A
C	C	D	A	B
D	D	A	B	C

Table 5

	<u>A</u>	<u>B</u>	<u>C</u>	<u>D</u>
A	A	B	C	D
B	B	A	D	C
C	C	D	A	B
D	D	C	B	A

We have seen these tables before and noted that they are essentially different.

```
G5>> ORDERS    for Group Number 4
```

```
Group number 4 of Order 4
```

```
1 elements of order 1:  A
```

```
1 elements of order 2:  C
```

```
2 elements of order 4:  B D
```

```
G4>> ORDERS    for Group Number 5
```

```
Group number 5 of Order 4
```

```
1 elements of order 1:  A
```

```
3 elements of order 2:  B C D
```

```
0 elements of order 4:
```

Notice that the number of elements of various orders are different. Thus the tables cannot be obtained from each other by renaming the elements.

Theorem:

If two groups are isomorphic (i.e. they give the same table if the elements are suitably named) then they must have the same number of elements of each order.

Question:

Is the converse true? If two groups have the same number of elements of each order, must the groups be isomorphic?

Hmm

Great Question!

Hmm

Great Question!
Keep it in mind.

G1>> CHART Order of Groups (1-32 or 0) Number 0

Order Groups of that order

1 1 * = non-abelian

2 2

3 3

4 4 5

5 6

6 7 8*

7 9

8 10 11 12 13* 14*

9 15 16

10 17 18*

11 19

12 20 21 22* 23* 24*

13 25

14 26 27*

15 28

16 29 30 31 32 33 34* 35* 36* 37* 38* 39* 40* 41* 42*

1
2
3
4
5
6
7
8

Orders of
Groups

1
2
3
4 5
6
7 8*
9
10 11 12 13* 14*

Group Numbers

1
2
3
4
5
6
7
8

Orders of
Groups

1
2 Only one group
3 of order 1,2,3
4 5
6
7 8*
9
10 11 12 13* 14*

Group Numbers

1
2
3
4
5
6
7
8

Orders of
Groups

1
2
3
4 Two groups of
5 order 4
6
7 8*
9
10 11 12 13* 14*

Group Numbers

Theorem: For each n there is a group of order n

Proof:

Z_n

Question:

For which n is there only one group of order n ?

G1>> CHART Order of Groups (1-32 or 0) Number 0

Order Groups of that order

1 1 * = non-abelian

2 2

3 3

4 4 5

5 6

6 7 8*

7 9

8 10 11 12 13* 14*

9 15 16

10 17 18*

11 19

12 20 21 22* 23* 24*

13 25

14 26 27*

15 28

16 29 30 31 32 33 34* 35* 36* 37* 38* 39* 40* 41* 42*

For groups of order 1-16, there is only one group of order n for the following n :

1, 2, 3, 5, 7, 11, 13, 15

Conjecture:

If n is prime, there is only one group of order n (up to isomorphism).

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Converse:

If there is only one group of order n , n is prime.

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If there is only one group of order n , n is prime.

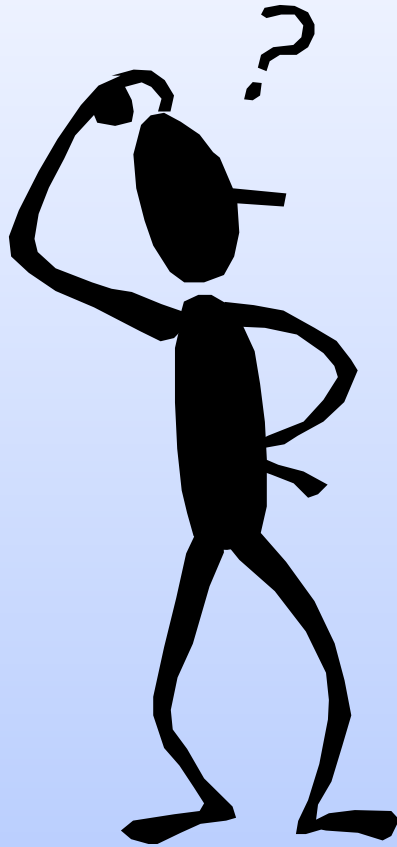
This false. $n=15$ is a counter-example.

Note: This fact may well have been used in making Groups32. We really need an independent proof. How would you show there can only be one group of a certain order?

G1>> CHART Order of Groups (1-32 or 0) Number 6
7 8*

There are 2 Groups of order 6
1 abelian and 1 non-abelian

What are the two groups of order 6?



Well, actually
I was waiting
for **you** to tell
me!

At this point:

You should know how many groups* there are of orders 1..7 and what they are.

* Up to isomorphism

Subgroups

G1>> **TABLE** for Group 4

	<u>A</u>	<u>B</u>	<u>C</u>	<u>D</u>
A	A	B	C	D
B	B	C	D	A
C	C	D	A	B
D	D	A	B	C

G4>> **ORDERS** for Group 4

Group number 4 of Order 4

1 elements of order 1: A

1 elements of order 2: C

2 elements of order 4: B D

G4>> **SUBGROUPS** of Group Number 4

* = Normal subgroup

Generators	Subgroup
0 { }	*{ A }
1 { C }	*{ A C }
2 { B }	*{ A B C D }

G4>> **TABLE** for Group **5**

	<u>A</u>	<u>B</u>	<u>C</u>	<u>D</u>
A	A	B	C	D
B	B	A	D	C
C	C	D	A	B
D	D	C	B	A

G5>> **ORDERS** for Group **5**

Group number 5 of Order 4

1 elements of order 1: A

3 elements of order 2: B C D

0 elements of order 4:

G5>> **SUBGROUPS** of Group Number **5**

* = Normal subgroup

Generators	Subgroup
0 { }	*{ A }
1 { B }	*{ A B }
2 { C }	*{ A C }
3 { D }	*{ A D }
4 { B C }	*{ A B C D }

Permutations

G5>> PERMGRPS

CREATE
INSTALL
X

ELEMENTS
MAIN

HELP
MULTIPLY

INFO
QUIT

Switching to PermGrps gives a new menu.

PERM>> MULTIPLY

Subgroup of S_n -- what is n ? Number 3

Put in a product of cycles.

End with a blank line

Cycles (1 2)(1 3) = (1 2 3)

Cycles (1 3)(1 2) = (1 3 2)

Cycles

The Permgroups package uses cycle notation. It is now set to multiply from left to right.

PERM>> CREATE

Subgroup of S_n -- what is n ? Number 3

Put in generators as product of cycles.

End with a blank line

Generator (1 2 3)

Generator (1 2)

Generator

Group is of order 6

A ()

B (2 3)

C (1 2)

D (1 2 3)

E (1 3 2)

F (1 3)

The smallest subgroup of S_3 which contains (1 2 3) and (1 2) is S_3 itself.

PERM>> **INSTALL**

Install as table k (1..5) Number **1**

PERM>> **MAIN**

CENTER	CENTRALIZER	CHART	CONJ-CLS
COSETS	EVALUATE	EXAMPLES	GENERATE
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TABLE	X		

Any group you create can be installed to replace one of the groups 1-5 in the main program. This allows you to play with it using the main commands. MAIN returns to the main program.

G1>> ORDERS for Group Number 1

Group number 1 of Order 6

1 elements of order 1: A

3 elements of order 2: B C F

2 elements of order 3: D E

0 elements of order 6:

It is useful to keep in mind how letters are assigned to permutations:

A ()	B (2 3)	C (1 2)
D (1 2 3)	E (1 3 2)	F (1 3)

G1>> SUBGROUPS of Group Number 1

* = Normal subgroup

Generators	Subgroup
0 { }	*{ A }
1 { B }	{ A B }
2 { C }	{ A C }
3 { F }	{ A F }
4 { D }	*{ A D E }
5 { B C }	*{ A B C D E F }

It is useful to keep in mind how letters are assigned to permutations:

A ()	B (2 3)	C (1 2)
D (1 2 3)	E (1 3 2)	F (1 3)

The IDENTIFY command will tell you which numbered group corresponds to the group you just generated. It tells us that the permutation group (installed as group 1) is the same as group 8.

```
G1>> IDENTIFY Group Number 1
```

```
G32 number:8 Order:6
```

```
GAP SmallGroup(6,1)
```

```
Generators:
```

```
B = (1 2)(3 4)(5 6)
```

```
C = (1 3)(2 5)(4 6)
```

This shows that group 8 is S_3 . The other group of order 6 is Z_6 . The CHART command shows two groups of order 6: group 7 is abelian; group 8 is non-abelian.

Conclusion

The commands in *Groups32* cover topics that occur in most introductory courses in abstract algebra. Once you have learned about some of these topics, it should be easy to discover how to use *Groups32* to do computations.

Many groups have "common names" like Z_6 and S_3 . Keep a list of common names for numbered groups as you identify them.