Math 103A – Winter,2001
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Cayley Tables

A (binary) operation on a finite set can be represented by a table. This is a square grid with one row and one column for each element in the set. The grid is filled in so that the element in the row belonging to x and the column belonging to y is x*y. Example:

This is a table for a binary operation on the set {A,B,C,D,E,F}

	A	B	C	D	E	F
A	A	B	C	D	E	F
B	B	C	A	F	D	E
C	C	A	B	E	F	D
D	D	E	F	A	B	C
E	E	F	D	C	A	B
F	F	D	E	B	C	A

Notice that B*E = D while E*B = F so this operation is not commutative.

1. Does this operation have an identity element? What is it?
2. Does every element have an inverse? If so, list the inverse of each element.
3. To check that the table is associative, you would have to check that (x*y)*z = x*(y*z) for any substitution of set elements for x,y,z. Try a few of these yourself – estimate how long it would take for you to check associativity. OR actually check the associativity and time yourself.
4. This is a 6 x 6 table. On the basis of your answer to 3, estimate how long it would take for you to check a 12 x 12 table. (NOTE: the answer is NOT twice as long)
5. A thought for the future: is there any way to verify that an operation is associative without checking case by case?
6. We say that the element x commutes with the element y if x*y = y*x. In the operation above we have noticed that B does not commute with E. Does B commute with other elements? List the elements that B commutes with.

   Here you must explain your answer. (for example, could we make B*B = B? If not, why not.)

7. Here are three tables for groups of order 4. In these tables, A is the identity. You will have to believe (or verify) associativity. You can verify the existence of inverses by inspection.

table 1

	A	B	C	D
A	A	B	C	D
B	B	C	D	A
C	C	D	A	B
D	D	A	B	C

table 2

	A	B	C	D
A	A	B	C	D
B	B	A	D	C
C	C	D	A	B
D	D	C	B	A

table 3

	A	B	C	D
A	A	B	C	D
B	B	A	D	C
C	C	D	B	A
D	D	C	A	B

Take table 1 and replace B by C and C by B. You will not, of course, get a table with the elements in the order A,B,C,D. So rearrange the rows and columns (making sure not to disturb the operation). [You should get table 3]

Table 1 and table 3 do not represent the same operation – however they differ only in the names given to the elements. Table 3 can be obtained from table 1 by changing the name of B to C and of C to B. The mapping $A \to A$, $B \to C$, $C \to B$, $D \to D$ transforms table 1 to table 3.

8. Give a convincing reason to show that, no matter how you map {A,B,C,D} to itself, you cannot transform table 1 to table 2. The binary operations defined by table 1 and table 2 are essentially different.

If there is a one-to-one correspondence between the set of elements in two tables which transforms one table to the other, we say the tables are isomorphic. Tables 1 and 3 are isomorphic – but table 2 is not isomorphic to either of the others. If two groups have isomorphic tables, they have the same "structural" properties.