Math 103A

Midterm 2 Answers

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Part I - True/False

1. circle T (F) If G has order n, it can have a subgroup H of order n-1 where $n > 2$.

2. circle T (F) G is a group. We have $(ab)^n = a^n b^n$ for all $a,b \in G$ and $n>0$.

3. circle (T) F If $x^2 = e$ for all x in G then G is abelian.

4. circle T (F) If H1 and H2 are subgroups of the same order in an abelian group G then H1 = H2.

5. circle T (F) If H is a normal subgroup of G and if H and G/H are abelian, then G is abelian.

6. circle T (F) A normal subgroup is always abelian

7. circle T (F) A subgroup which is abelian is always normal

8. circle (T) F If G is abelian then every subgroup of G is normal

9. circle T (F) A subgroup of prime order is always normal

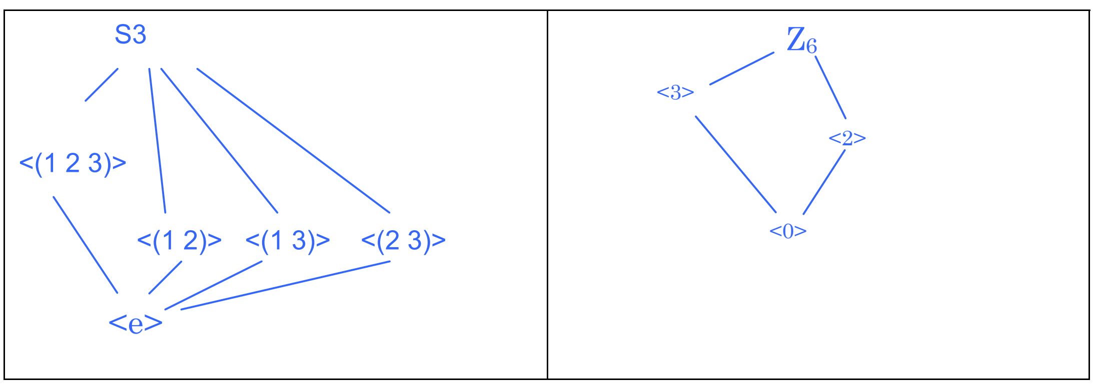
10. circle T (F) A subgroup of prime index is always normal

Part II - longer problems

1. Let G be a group and H a subgroup. Prove the following statements about right cosets:
   a. H is a right coset of H
   b. Any element of G is in some right coset of H.
   c. The right cosets are disjoint in the sense that $\forall$ a,b $\in$ G either Ha = Hb or Ha $\cap$ Hb = $\varnothing$ (empty set)

   a. H = He , the right coset of the identity element
   b. a = ea $\in$ Ha
   c. If c $\in$ Ha $\cap$ Hb we have c = ha = kb for h,k $\in$ H so a = $h^{-1}kb$ $\in$ Hb and therefore Ha $\subset$ Hb. Similarly Hb $\subset$ Ha so Ha = Hb.

2. a. Find all subgroups of S_3 and construct the subgroup lattice.
   b. Find all subgroups of $\mathbb{Z}_6$ and construct the subgroup lattice.



3. H is a subgroup of a group G. Show that if | H | = ½ |G| then H is normal

A subgroup is normal if and only if every right coset is a left coset. In this case there are only two cosets (left or right) one of which is H = eH = He. If a is not in H then aH = Ha consists of the elements of G which are not in H. So every left coset is a right coset.

4. Let G be a group and $a \in G$. The centralizer of a is the set
$C(a) = \{x \in G \mid xa = ax\}$

   a. Show that C(a) is a subgroup of G

   b. Compute C(a) if $G = S_3$ and $a = (1,2,3)$

   c. Compute C(a) if $G = S_3$ and $a = (1,2)$

a. If $x,y \in C(a)$ then $xya = x(ay) = (xa)y = axy$ -- so C(a) is closed

multiply $ax = xa$ on the left and right by x^{-1} to obtain $x^{-1} a = ax^{-1}$ so C(a) is closed under inverse

b. We will use Lagrange's theorem. | C(a) | divides 6, which is the order of the group. Note that e, a = (1 2 3) , a^2 = (1 3 2) all commute with a So C(a) has at least 3 elements. However (1 2) does not commute with (1 2 3). So C(a) cannot be of order 6 – so C(a) = <(1 2 3)>

c. The same argument applies when a = (1 2). The subgroup C(a) contains <(1 2)> but cannot be of order 6 since (1 2 3) does not commute with (1 2)