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# A counterexample to sparse removal

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**Abstract**

The *Turán number* of a graph $H$, denoted $\mathrm{ex}(n, H)$, is the maximum number of edges in an $n$-vertex graph with no subgraph isomorphic to $H$. Solymosi [17] conjectured that if $H$ is any graph and $\mathrm{ex}(n, H) = O(n^\alpha)$ where $\alpha > 1$, then any $n$-vertex graph with the property that each edge lies in exactly one copy of $H$ has $o(n^\alpha)$ edges. This can be viewed as conjecturing a possible extension of the removal lemma to sparse graphs, and is well-known to be true when $H$ is a non-bipartite graph, in particular when $H$ is a triangle, due to Ruzsa and Szemerédi [16]. Using Sidon sets we exhibit infinitely many bipartite graphs $H$ for which the conjecture is false.

## 1 Introduction

The Removal Lemma [6, 5] states that if $(G_n)_{n\in\mathbb{N}}$ is a sequence of graphs where $G_n$ has $n$ vertices, and $H$ is a $k$-vertex graph such that $G_n$ contains $o(n^k)$ subgraphs isomorphic to $H$, then $o(n^2)$ edges may be deleted from $G_n$ to obtain an $H$-free graph. Ruzsa and Szemerédi [16] established this result in the case that $H$ is a triangle as a consequence of Szemerédi's Regularity Lemma [18], and then applied the removal lemma to give a weak form of Roth's Theorem [14] on three-term arithmetic progressions. The removal lemma is a central tool in extremal combinatorics with many applications [5].

The removal lemma is not effective if $G_n$ has $o(n^2)$ edges. In particular, if $H$ is a $k$-vertex bipartite graph and $G_n$ contains $o(n^k)$ copies of $H$, then it is a consequence of extremal graph theory that $G_n$ has $o(n^2)$ edges. The *Turán number* $\mathrm{ex}(n, H)$ is the maximum number of edges in an $n$-vertex graph not containing $H$. When $H$ is a bipartite graph, the Kövari-Sós-Turán Theorem [11] immediately shows $\mathrm{ex}(n, H) = O(n^{2-1/k})$, and more precise results are available [1, 9] dependent on the finer structure of $H$. Solymosi [17] conjectured an extension of the removal lemma in this regime as a function of the Turán number of $H$ as follows. The *exponent* of a graph $H$, when it exists, is a real number $\alpha$ such that $\mathrm{ex}(n, H) = \Theta(n^\alpha)$ as $n \to \infty$. Erdős and Simonovits [8] conjectured that every graph has an exponent, but this

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conjecture remains open. In the case that $H$ is not bipartite the exponent is 2, whereas in the bipartite case the exponent is generally not known. Solymosi's conjecture is as follows:

**Conjecture 1.** *If $H$ is a graph with exponent $\alpha$, then any $n$-vertex graph in which every edge is in exactly one copy of $H$ has $o(n^\alpha)$ edges.*

In the case $H = C_4$ i.e. $H$ is a quadrilateral, Solymósi [17] conjectured that if an $n$ vertex graph is a union of $\Theta(n^{3/2})$ edge-disjoint quadrilaterals, then the graph contains $\Omega(n^2)$ quadrilaterals. By the removal lemma, Conjecture 1 is true for non-bipartite graphs $H$, so the conjecture is interesting for bipartite graphs.

### 1.1 Main Result

Our main result shows that there are infinitely many bipartite graphs $H$ for which Conjecture 1 is false. Let $H_k$ be the graph with vertex set $V(H_k) = \{1, 2, \ldots, 2k\}$ and edge set $E(H_k) = \{1i, 2i, 3j, 4j : 3 \leq i \leq k+2 < j \leq 2k\}$ – see Figure 1.

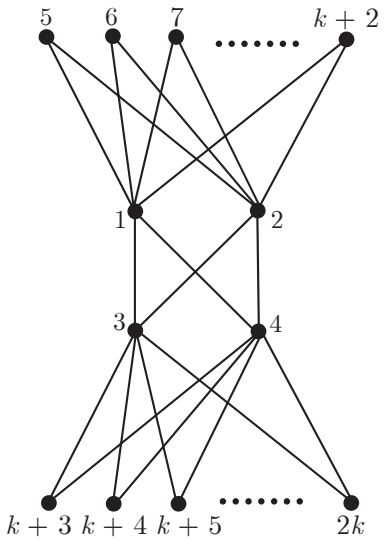


Figure 1: The bipartite graph $H_k$.

We shall prove that $H_k$ has exponent $\frac{3}{2}$ for all $k \geq 3$ (Section 4), and then the following theorem shows that the graphs $H_k$ for $k \geq 5$ give counterexamples to Conjecture 1.

**Theorem 1.1.** *Let $k \geq 5$ and let $P$ be the set of primes $p \equiv 1$ mod 4. There exists a sequence of graphs $(G_p)_{p \in P}$ such that $G_p$ has $n = 2kp^2$ vertices, $\Theta(n^{3/2})$ edges, and every edge of every $G_p$ is contained in exactly one copy of $H_k$.*

The proof of Theorem 1.1 is based on a construction described in Section 2 involving simple field arithmetic and the quadratic character of $\mathbb{F}_p$. The proof is given in Sections 2 – 4. The case that $H$ is a quadrilateral in Conjecture 1 remains open.

## 2 Outline of the Proof of Theorem 1.1

The proof of Theorem 1.1 is achieved in three steps. Before we describe these steps, we introduce some notation. Throughout this section, $p$ is a prime, $k$ is a positive integer, and $\mathbb{F}_p$ denotes the finite field of order $p$. Let $\chi$ denote the quadratic character of $\mathbb{F}_p$, namely $\chi(x) = 1$ if $x$ is a non-zero quadratic residue, $\chi(x) = -1$ if $x$ is not a quadratic residue, and $\chi(0) = 0$. The graph $H_k$ has vertex set $[2k] := \{1, 2, \ldots, 2k\}$ and edge set

$$E(H_k) := \{1i, 2i, 3j, 4j : 3 \leq i \leq k+2 < j \leq 2k\}.$$

Let $V(G)$ and $E(G)$ denote the vertex set and edge set of a graph $G$. If $G_1$ and $G_2$ are edge-disjoint graphs, then we write $G = G_1 \oplus G_2$ if $V(G) = V(G_1) \cup V(G_2)$ and $E(G) = E(G_1) \cup E(G_2)$.

### 2.1 Step I. Construction of graphs $G_{\Gamma,\Lambda,S}(H)$.

Let $\Gamma$ be a finite abelian group and $S \subseteq \Gamma$. Let $H$ be an arbitrary graph with vertex set $[k]$, edge set $E$, and let $\Lambda = \{\lambda_1, \lambda_2, \ldots, \lambda_k\} \subset \mathbb{Z}$. For $i \in [k]$ let $X_i = \Gamma \times \{i\}$. For $ij \in E$ with $i < j$, let $G_{\Gamma,\Lambda,S}(ij)$ be the bipartite graph with parts $X_i$ and $X_j$, where $x \in X_i$ is adjacent to $y \in X_j$ whenever there exists $a \in S$ such that $y = x + (\lambda_j - \lambda_i)a$. Finally, define the following $k$-partite graph with parts $X_1, X_2, \ldots, X_k$:

$$G_{\Gamma,\Lambda,S}(H) = \bigcup_{ij \in E} G_{\Gamma,\Lambda,S}(ij).$$

A key observation is that $G_{\Gamma,\Lambda,S}(H)$ is built up from edge-disjoint copies of $H$. The following is proved in Section 3:

**Proposition 1.** *Let $k \in \mathbb{N}$ and let $H$ be a $k$-vertex graph. For any finite abelian group $\Gamma$, $S \subseteq \Gamma$, and $\Lambda \subset \mathbb{Z}$, the graph $G_{\Gamma,\Lambda,S}(H)$ has $k|\Gamma|$ vertices, $|E||\Gamma||S|$ edges, and there exist induced subgraphs $H(1), H(2), \ldots, H(|\Gamma||S|)$ of $G_{\Gamma,\Lambda,S}(H)$, each isomorphic to $H$, such that*

$$G_{\Gamma,\Lambda,S}(H) = \bigoplus_{i=1}^{|\Gamma||S|} H(i).$$

The proof of Theorem 1.1 involves a suitable choice of the group $\Gamma$, the set $S$, and the set $\Lambda \subset \mathbb{Z}$. The graphs $G_{\Gamma,\Lambda,S}(H)$ may be useful as constructions for other extremal graph theory problems.

### 2.2 Step II. The choice of $\Gamma$, $S$, and $\Lambda$.

Let $\Gamma = \mathbb{F}_p^2$ and $S = \{(a, a^2) : a \in \mathbb{F}_p^*\}$ where $p \equiv 1 \bmod 4$ is prime. We aim to show that there exists a choice of $\Lambda = \{\lambda_1, \ldots, \lambda_k\}$ such that every edge of $G_p = G_{\Gamma,\Lambda,S}(H_k)$ is in exactly one copy of $H_k$, which is the heart of Theorem 1.1. It is convenient to let $\lambda_{ij} = \lambda_j - \lambda_i$. After some arithmetic preparations in Section 4.1, we prove the following in Section 4.2.

**Proposition 2.** *Let $p \equiv 1$ mod 4 be a prime, $\Gamma = \mathbb{F}_p^2$, and $S = \{(a, a^2) : a \in \mathbb{F}_p^*\}$. Suppose $\Lambda = \{\lambda_i : i \in [2k]\}$ and*

1. $\lambda_1 = 0$, $\lambda_2 = 1$.
2. *For $i \in \{3, 4\}$ and $j \in \{1, 2\}$, $\chi(\lambda_{1i}\lambda_{i2}) = \chi(\lambda_{3j}\lambda_{j4}) = -1$.*
3. *For $5 \leq i \leq k + 2 < j \leq 2k$, we have $\chi(\lambda_{1i}\lambda_{i2}) = \chi(\lambda_{3j}\lambda_{j4}) = 1$.*

*Then every edge of $G_p = G_{\Gamma,\Lambda,S}(H_k)$ is contained in exactly one copy of $H_k$.*

Note that $\chi(\lambda_{ij}) = \chi(-\lambda_{ji}) = \chi(\lambda_{ji})$ since $p \equiv 1$ mod 4. A set $\Lambda$ satisfying the conditions in Proposition 2 is called *suitable*. We prove the following in Section 5:

**Proposition 3.** *If $k \geq 5$ and $p \geq 4k + 3$, then there exist a suitable set $\Lambda \subset \mathbb{F}_p$.*

If $\Gamma = \mathbb{F}_p^2$ and $S = \{(a, a^2) : a \in \mathbb{F}_p^*\}$, then it is not hard to show $G_{\Gamma,S,\Lambda}(C_4)$ contains $\Theta(n^2)$ quadrilaterals for any set $\Lambda$, so $G_{\Gamma,S,\Lambda}(C_4)$ cannot be used as a counterexample to Conjecture 1 when $H = C_4$.

### 2.3 Step III. The Turán Number for $H_k$.

It remains to show $\mathrm{ex}(n, H_k) = \Theta(n^{3/2})$, which is the final step in the proof of Theorem 1.1. A counting argument will be used to determine the order of magnitude of $\mathrm{ex}(n, H_k)$.

**Proposition 4.** *For any integer $k \geq 3$, $ex(n, H_k) = \Theta(n^{3/2})$.*

This proposition is proved in Section 6. Since $K_{2,k} \subset H_k$, the results of Füredi [10] show

$$\mathrm{ex}(n, H_k) \geq \mathrm{ex}(n, K_{2,k}) \geq \frac{1}{2}\sqrt{k-1}n^{3/2} + o(n^{3/2})$$

and this establishes the lower bound in Proposition 4.

## 3 Proof of Proposition 1

It follows from the definition of $G_{\Gamma,\Lambda,S}(H)$ that $G_{\Gamma,\Lambda,S}(H)$ has $k|\Gamma|$ vertices. We now prove that $G_{\Gamma,\Lambda,S}(H)$ is a union of edge-disjoint copies of $H$. We observe that $H$ appears naturally as a subgraph of $G_{\Gamma,\Lambda,S}(H)$ in the following manner: for $v \in \Gamma$ and $a \in S$, the set $\{v_i = v + \lambda_i a : i \in [k]\}$ induces a subgraph $H(v, a)$ isomorphic to $H$, since $v_i v_j \in G_{\Gamma,\Lambda,S}(H)$ if and only if $ij \in E(H)$. Also, if $(x, i), (y, j) \in H(v, a)$ and $(x, i), (y, j) \in H(w, b)$, then

$$x = v + \lambda_i a = w + \lambda_i b \quad y = v + \lambda_j a = w + \lambda_j b$$

and therefore $\lambda_i(a - b) = \lambda_j(a - b)$ which means $a = b$ and so $v = w$. Therefore, no two subgraphs $H(v, a)$ share any pair of vertices. We conclude

$$G_{\Gamma,\Lambda,S}(H) = \bigoplus_{(v,a)\in\Gamma\times S} H(v, a).$$

In particular, $G_{\Gamma,\Lambda,S}(H)$ has $|E||\Gamma||S|$ edges. This proves Proposition 1. □

# 4 Proof of Proposition 2

This section is split into two parts. First we describe the interaction between quadrilaterals in $G_p(H) := G_{\Gamma,\Lambda,S}(H)$ for general $H$, $\Gamma = \mathbb{F}_p^2$, $\Lambda \subset \mathbb{F}_p$ and $S = \{(a, a^2) : a \in \mathbb{F}_p^*\}$. We then show that if the conditions of Proposition 2 are satisfied, then every edge of $G_p = G_p(H_k)$ is in exactly one copy of $H_k$.

## 4.1 Quadrilaterals in $G_{\Gamma,\Lambda,S}(H)$.

Throughout this subsection, $H$ is a graph, $G_p(H) = G_{\Gamma,\Lambda,S}(H)$ where $\Gamma = \mathbb{F}_p^2$, $\Lambda \subset \mathbb{F}_p$ and $S = \{(a, a^2) : a \in \mathbb{F}_p^*\}$. The following simple but key arithmetic lemma is due to Ruzsa [15].

**Lemma 1.** *Let $\alpha, \beta, \gamma, \delta, a, b, c, d \in \mathbb{F}_p^*$ where $\alpha + \beta = \gamma + \delta \neq 0$. If*

$$\alpha(a, a^2) + \beta(b, b^2) = \gamma(c, c^2) + \delta(d, d^2),$$

*then $\alpha\beta(a - b)^2 = \gamma\delta(c - d)^2$.*

*Proof.* Multiply $\alpha a^2 + \beta b^2 = \gamma c^2 + \delta d^2$ by $\alpha + \beta = \gamma + \delta$ to get

$$\alpha^2 a^2 + \alpha\beta(a^2 + b^2) + \beta^2 b^2 = \gamma^2 c^2 + \gamma\delta(c^2 + d^2) + \delta^2 d^2.$$

Subtracting the square of $\alpha a + \beta b = \gamma c + \delta d$ gives $\alpha\beta(a - b)^2 = \gamma\delta(c - d)^2$. □

This lemma has a number of consequences relative to the distribution of quadrilaterals in $G(ij) := G_{\Gamma,\Lambda,S}(ij)$ for $ij \in H$. For $hi, ij \in E(H)$, let $G(hij) = G(hi) \cup G(ij)$.

**Lemma 2.** *For any edge $ij \in E(H)$, the graph $G(ij)$ is $C_4$-free.*

*Proof.* Consider a quadrilateral $(x, i)(y, j)(z, i)(w, j) \subset G(ij)$. For some $a, b, c, d \in \mathbb{F}_p^*$,

$$y = x + \lambda_{ij}(a, a^2) = z + \lambda_{ij}(b, b^2) \quad w = x + \lambda_{ij}(c, c^2) = z + \lambda_{ij}(d, d^2).$$

Canceling out $x, y, z$ and $w$ and dividing by $\lambda_{ij}$, we obtain in each component $a + b = c + d$ and $a^2 + b^2 = c^2 + d^2$. By Lemma 1, $(a - b)^2 = (c - d)^2$ and so $a - b = c - d$ or $a - b = d - c$. In either case, together with $a + b = c + d$ we find $\{a, b\} = \{c, d\}$. But then $(x, i) = (z, i)$ or $(y, j) = (w, j)$, a contradiction. Therefore $G(ij)$ has no $C_4$. □

**Lemma 3.** *For any edges $hi, ij \in E(H)$, the graph $G(hij)$ is $K_{2,3}$-free.*

*Proof.* Suppose $G(hij)$ contains a $K_{2,3}$. By Lemma 2, $G(hi)$ and $G(ij)$ are $C_4$-free, and so the three vertices of degree two in the $K_{2,3}$ must be in $X_i$, say $(z_1, i), (z_2, i), (z_3, i)$, and the other two vertices are $(x, h) \in X_h$ and $(y, j) \in X_j$. By definition, for some $a_1, a_2, a_3 \in \mathbb{F}_p^*$ and $b_1, b_2, b_3 \in \mathbb{F}_p^*$,

$$y = x + \lambda_{hi}(a_r, a_r^2) + \lambda_{ij}(b_r, b_r^2)$$

for $r \in \{1,2,3\}$. Note that the $a_r$ are distinct and the $b_r$ are distinct, for if $a_r = a_s$ or $b_r = b_s$ for some $r \neq s$, then $z_r = z_s$ and so $(z_r, i) = (z_s, i)$, a contradiction. On the other hand, by Lemma 1, $(a_r - b_r)^2 = (a_s - b_s)^2$ for all $r, s \in \{1,2,3\}$. Taking square roots, some pair of square roots has the same sign, namely for some $r \neq s$, we have $a_r - b_r = a_s - b_s$. For all $r, s$, we also have

$$\lambda_{hi} a_r + \lambda_{ij} b_r = \lambda_{hi} a_s + \lambda_{ij} b_s.$$

Subtracting $\lambda_{hi}(a_r - b_r) = \lambda_{hi}(a_s - b_s)$ from this equation, we obtain $\lambda_{hj} b_r = \lambda_{hj} b_s$. Therefore, $b_r = b_s$ and $(z_r, i) = (z_s, i)$ which is a contradiction. □

**Lemma 4.** *If $C = (x,i)(y,j)(z,k)(w,l)$ is a quadrilateral in $G_p(H)$ and $(x,i)(y,j)(z,k) \subset H(v,a)$, then $C \subset H(v,a)$.*

*Proof.* By definition of $G_p(H)$, there exist $a, c, d \in \mathbb{F}_p^*$ such that

$$y = x + \lambda_{ij}(a, a^2) \quad z = y + \lambda_{jk}(a, a^2) \quad w = z + \lambda_{kl}(c, c^2) \quad x = w + \lambda_{li}(d, d^2).$$

This implies

$$\lambda_{ij}(a, a^2) + \lambda_{jk}(a, a^2) = \lambda_{lk}(c, c^2) + \lambda_{il}(d, d^2).$$

By Lemma 1 with $\alpha = \lambda_{ij}$, $\beta = \lambda_{jk}$, $\gamma = \lambda_{lk}$, and $\delta = \lambda_{il}$, noting $\alpha + \beta = \gamma + \delta$, we have

$$\gamma\delta(c - d)^2 = 0$$

and therefore $c = d$. Now

$$\lambda_{ik} a = \lambda_{ij} a + \lambda_{jk} a = \lambda_{lk} c + \lambda_{il} d = \lambda_{ik} c$$

so we conclude $a = c = d$. In particular, since $z = v + \lambda_{0k}(a, a^2)$,

$$w = z + \lambda_{kl}(a, a^2) = v + \lambda_{0k}(a, a^2) + \lambda_{kl}(a, a^2) = v + \lambda_{0l}(a, a^2)$$

and we conclude $C \subset H(v,a)$. □

**Lemma 5.** *If $C = (x,i)(y,j)(z,k)(w,l)$ is a quadrilateral in $G_p(H)$ and $\chi(\lambda_{ij}\lambda_{jk}\lambda_{kl}\lambda_{li}) = -1$ then $C \subset H(v,a)$ for some $(v,a) \in \mathbb{F}_p^2 \times S$.*

*Proof.* By definition of $G_p(H)$, there exist $a, b, c, d \in \mathbb{F}_p^*$ such that

$$\lambda_{ij}(a, a^2) + \lambda_{jk}(c, c^2) = \lambda_{il}(d, d^2) + \lambda_{lk}(b, b^2).$$

By Lemma 1,

$$\lambda_{ij}\lambda_{jk}(a - c)^2 = \lambda_{il}\lambda_{lk}(d - b)^2.$$

Since $\chi(\lambda_{ij}\lambda_{jk}\lambda_{il}\lambda_{lk}) = \chi(\lambda_{ij}\lambda_{jk}\lambda_{kl}\lambda_{li}) = -1$, we conclude $a = c$ and $b = d$. The equation $\lambda_{ij} a + \lambda_{jk} c = \lambda_{il} d + \lambda_{lk} b$ reduces to $\lambda_{ik} a = \lambda_{ik} d$ and we conclude $a = b = c = d$. Letting $v = x - \lambda_{0i}(a, a^2)$, we have $C \subset H(v,a)$. □

### 4.2 Proof of Proposition 2

Suppose $F \subset G_p = G_{\Gamma,\Lambda,S}(H_k)$ is isomorphic to $H_k$ and let $\phi : V(H_k) \to V(F)$ be an isomorphism. We aim to show that $F = H(v,a)$ for some $v \in \mathbb{F}_p$ and $a \in S$ by finding, via Lemma 5, a quadrilateral $C^* \subset H_k$ with $\phi(C^*) \subset H(v,a)$. This is the point where we make heavy use of the last two conditions in Proposition 2. Let $\mathfrak{c} : V(H_k) \to \Lambda$ be the proper vertex-coloring of $H_k$ given by $\mathfrak{c}(x) = \lambda_i$ if and only if $\phi(x) \in X_i$. Let $[m,n] := \{m, m+1, m+2, \ldots, n\}$.

**Claim 1.** *Each quadrilateral in $H_k$ is colored with at least three colors, and each $K_{2,3}$ in $H_k$ is colored with at least four colors. Furthermore,*

$$\{\mathfrak{c}(1), \mathfrak{c}(2), \mathfrak{c}(3), \mathfrak{c}(4)\} \subset \{\lambda_1, \lambda_2, \lambda_3, \lambda_4\}.$$

*Proof of Claim 1.* The first statement is an immediate consequence of Lemmas 2 and 3. Consider the vertex $1 \in V(H_k)$ (see Figure 1). If $\mathfrak{c}(1) \in \{\lambda_5, \lambda_6, \ldots, \lambda_{k+2}\}$ then each neighbor of 1 is assigned color $\lambda_1$ or color $\lambda_2$. This follows from the fact that if $xy \in E(H_k)$ and $\mathfrak{c}(x) = \lambda_i$ and $\mathfrak{c}(y) = \lambda_j$, then $ij \in E(H_k)$. The neighbors of 1 are also neighbors of 2 so that $\mathfrak{c}$ assigns one color to at least three common neighbors of 1 and 2. This is impossible by the first part of the claim. A similar argument shows that $\mathfrak{c}(1) \notin \{\lambda_{k+3}, \lambda_{k+4}, \ldots, \lambda_{2k}\}$ thus $\mathfrak{c}(1) \notin \{\lambda_5, \lambda_6, \ldots, \lambda_{2k}\}$. By symmetry, we must have $\mathfrak{c}(i) \notin \{\lambda_5, \lambda_6, \ldots, \lambda_{2k}\}$ for $1 \leq i \leq 4$ and so

$$\{\mathfrak{c}(1), \mathfrak{c}(2), \mathfrak{c}(3), \mathfrak{c}(4)\} \subset \{\lambda_1, \lambda_2, \lambda_3, \lambda_4\}.$$

This proves Claim 1. □

**Claim 2.** *There is a quadrilateral $C^* = ghij$ in $H_k$ such that*

$$\chi\left((\mathfrak{c}(g) - \mathfrak{c}(h))(\mathfrak{c}(h) - \mathfrak{c}(i))(\mathfrak{c}(i) - \mathfrak{c}(j))(\mathfrak{c}(j) - \mathfrak{c}(g))\right) = -1.$$

*Proof of Claim 2.* By Claim 1 there is a path $ghi \subset 1324$ such that $\mathfrak{c}(g), \mathfrak{c}(h), \mathfrak{c}(i)$ are distinct. Without loss of generality, assume that $g = 1$, $h = 3$, and $i = 2$. If we can find a $j \in [5, k+2]$ such that $\mathfrak{c}(j) \notin \{\lambda_1, \lambda_2, \lambda_3, \lambda_4\}$ then by the conditions of Proposition 2,

$$\chi(\lambda_{13}\lambda_{32}\lambda_{2j}\lambda_{j1}) = \chi(\lambda_{13}\lambda_{32}) \cdot \chi(\lambda_{2j}\lambda_{j1}) = -1$$

and so $132j$ is the required cycle. To find $j$, note that no two vertices in $[5, k+2]$ have color $\mathfrak{c}(3)$ otherwise those two vertices together with $1, 3$ and $2$ form a 3-colored $K_{2,3}$, contradicting Claim 1. Similarly, no two vertices in $[5, k+2]$ have color $\mathfrak{c}(4)$. Since $k \geq 5$, there are at least three vertices in $[5, k+2]$ and so one of them, say $j$, has a color not in $\{\lambda_1, \lambda_2, \lambda_3, \lambda_4\}$, as required. □

We now complete the proof of Proposition 2. By Lemma 5, there exists $v \in \mathbb{F}_p^2$ and $a \in S$ such that $\phi(C^*) \subset H(v,a)$. Given any edge $e$ of $F = \phi(H_k)$, there exist quadrilaterals $C^* = C^1, C^2, \ldots, C^r \subset H_k$ such that $e \in E(\phi(C^r))$, and $E(\phi(C^i)) \cap E(\phi(C^{i+1})) \neq \emptyset$ for

$i < r$. By Lemma 4, we inductively have $\phi(C^i) \subset H(v,a)$ for all $i \leq r$. In particular, $e \in E(H(v,a))$ and we conclude $F \subset H(v,a)$. Since $H(v,a)$ is an induced subgraph of $G_p(H_k)$ isomorphic to $F$, we conclude $F = H(v,a)$. □

# 5 Proof of Proposition 3

To build a suitable set $\Lambda \subset \mathbb{F}_p$, we use the following identity (see Theorem 5.48, [13]):

**Proposition 5.** *Let $f(x) = a_2x^2 + a_1x + a_0 \in \mathbb{F}_p[x]$ where $a_2 \neq 0$. If $a_1^2 - 4a_0a_2 \neq 0$ then*

$$\sum_{c \in \mathbb{F}_p} \chi(f(c)) = -\chi(a_2).$$

Let $k \geq 5$, $p \geq 4k+3$ be prime with $p \equiv 1 (\mathrm{mod}\ 4)$, and $\lambda_1 = 0$, $\lambda_2 = 1$. By Proposition 5,

$$\sum_{c \in \mathbb{F}_p} \chi(c^2 - c) = -1 \tag{1}$$

so there are least $\frac{p-3}{2} \geq 2k$ elements of $\mathbb{F}_p$ for which $\chi(c^2-c) = -1$. Let $\lambda_3$ be any one of them and observe that since $\chi(\lambda_3^2 - \lambda_3) = -1$, we have $\lambda_3 \neq 0$ and $\lambda_3 \neq 1$. Using the fact that $\chi(-1) = 1$,

$$-1 = \chi(\lambda_3^2 - \lambda_3) = \chi((0-\lambda_3)(\lambda_3 - 1)) = \chi((\lambda_1 - \lambda_3)(\lambda_3 - \lambda_2)).$$

Next we choose $\lambda_4$. Let $g(x) = (1 - \chi(x^2 - x))(1 - \chi(\lambda_3 x))$ and $X = \{c \in \mathbb{F}_p : g(c) = 4\}$. If $c \in X$ then $\chi(c^2 - c) = \chi(\lambda_3 c) = -1$. Suppose $c \in \mathbb{F}_p$ and $0 < g(c) \leq 2$. Then either $\chi(c^2 - c) = 0$ or $\chi(\lambda_3 c) = 0$ which means $c = 0$ or $c = 1$ thus

$$\Big| \sum_{c \in \mathbb{F}_p} g(c) - 4|X| \Big| \leq 2 \cdot 2. \tag{2}$$

Expanding $g(c)$ and using Proposition 5,

$$\sum_{c \in \mathbb{F}_p} g(c) = p + \sum_{c \in \mathbb{F}_p} \big( -\chi(c^2 - c) - \chi(\lambda_3)\chi(c) + \chi(c^2)\chi(\lambda_3)\chi(c-1) \big) = p + 1. \tag{3}$$

Here we have used the well known fact that $\sum_{c \in \mathbb{F}_p} \chi(c) = 0$. By (2) and (3),

$$|X| \geq \frac{p+1}{4} - 1$$

and since $\frac{p-3}{4} \geq 1$, the set $X$ is not empty. Choose $\lambda_4$ so that $-1 = \chi(\lambda_4(\lambda_4 - 1)) = \chi(\lambda_3\lambda_4)$. Then $\lambda_4$ is not equal to any of $\lambda_1, \lambda_2$, or $\lambda_3$. The relation $-1 = \chi(\lambda_4(\lambda_4 - 1))$ implies $-1 = \chi((\lambda_1 - \lambda_4)(\lambda_4 - \lambda_2))$ and the relation $-1 = \chi(\lambda_3\lambda_4)$ implies $-1 = \chi((\lambda_3 - \lambda_1)(\lambda_1 - \lambda_4))$. Furthermore

$$\begin{aligned} \chi((\lambda_3 - \lambda_2)(\lambda_2 - \lambda_4)) &= \chi((\lambda_3 - 1)(\lambda_4 - 1)) \\ &= \chi(\lambda_3(\lambda_3 - 1))\chi(\lambda_3\lambda_4)\chi(\lambda_4(\lambda_4 - 1)) \\ &= (-1)(-1)(-1) = -1 \end{aligned}$$

so that all of the requirements of Condition 2 are satisfied.

Choosing the remaining $\lambda_i$'s will be straightforward. By (1), there are at least $\frac{p-3}{2}$ elements $c \in \mathbb{F}_p$ for which $\chi(c^2 - c) = 1$. Since $\frac{p-3}{2} \geq 2k$ we can choose $\lambda_5, \lambda_6, \ldots, \lambda_{k+2}$ so that none of these are equal to $\lambda_1$, $\lambda_2$, $\lambda_3$ or $\lambda_4$, and $\chi((\lambda_1 - \lambda_i)(\lambda_i - \lambda_2)) = 1$ for all $4 < i \leq k+2$.

Let $f(x) = (\lambda_3 - x)(x - \lambda_4) = -x^2 + (\lambda_3 + \lambda_4)x - \lambda_3\lambda_4$. If $a_2 = -1$, $a_1 = \lambda_3 + \lambda_4$, and $a_0 = -\lambda_3\lambda_4$, then $a_1^2 - 4a_0a_2 = (\lambda_3 - \lambda_4)^2 \neq 0$. By Proposition 5,

$$\sum_{c \in \mathbb{F}_p} \chi(f(c)) = -\chi(-1) = -1$$

and there are at least $\frac{p-3}{2}$ elements $c$ for which $\chi(f(c)) = 1$. Since $\frac{p-3}{2} \geq 2k$ we can choose $\lambda_{k+3}, \lambda_{k+4}, \ldots, \lambda_{2k}$ so that all of $\lambda_1, \lambda_2, \ldots, \lambda_{2k}$ are distinct and for any $4 < i \leq k+2$,

$$\chi((\lambda_3 - \lambda_{i+k-2})(\lambda_{i+k-2} - \lambda_4)) = 1.$$

This completes the proof of Proposition 5. □

## 6 Proof of Proposition 4

The claim $\mathrm{ex}(n, H_k) = \Theta(n^{3/2})$ follows by proving $\mathrm{ex}(n, H_k) \leq kn^{3/2}$ for all $k \geq 3$ and large enough $n$. We follow the method of dependent random choice [9], but we do not optimize the constants in the upper bound we obtain for $\mathrm{ex}(n, H_k)$.

Let $G$ be an $H_k$-free $n$-vertex graph with average degree $d := 2|E(G)|/n$. It is sufficient to show $d \leq 2k\sqrt{n}$. Choose uniformly at random a pair of vertices $\{x_1, x_2\}$ in $G$ and let $S = N(x_1, x_2)$, the common neighborhood of $x_1$ and $x_2$. Given a pair of vertices $\{y_1, y_2\}$, let $d(y_1, y_2) = |N(y_1, y_2)|$. Let $X = |S|$ and let $Y$ be the number of $\{y_1, y_2\} \subset S$ with $d(y_1, y_2) \leq 2k$. If $X - Y \geq k + 2$, then there exist two vertices $\{x_1, x_2\} \subset V(G)$ with $d(x_1, x_2) > k + 2$ and some pair $\{y_1, y_2\} \subset N(x_1, x_2)$ with $d(y_1, y_2) > 2k$, and we easily find a copy of $H_k$ by mapping $\{1, 2\}$ to $\{x_1, x_2\}$ and $\{3, 4\}$ to $\{y_1, y_2\}$, a contradiction. So $X - Y \leq k + 1$.

On the other hand, using convexity of binomial coefficients,

$$\begin{aligned} k+1 &\geq \mathbb{E}(X - Y) \\ &\geq \sum_{y \in V(G)} \frac{\binom{d(y)}{2}}{\binom{n}{2}} - \binom{n}{2} \cdot \frac{\binom{2k}{2}}{\binom{n}{2}} \\ &\geq \frac{1}{\binom{n}{2}} n \binom{d}{2} - \binom{2k}{2} \\ &\geq \frac{d(d-1)}{n} - \binom{2k}{2}. \end{aligned}$$

It follows that if $n$ is large enough, then $d \leq 2k\sqrt{n}$. This proves Proposition 4. □

# 7 Concluding remarks

• A *Sidon set* in a finite abelian group $\Gamma$ is a set $S \subset \Gamma$ such that if $a+b=c+d$ with $a,b,c,d \in S$, then $\{a,b\}=\{c,d\}$. A generalization of Sidon sets was given in [12]. Consider all equations $\alpha a+\beta b=\gamma c+\delta d$ where $\alpha+\beta=\gamma+\delta$ and $1 \le \alpha,\beta,|\gamma|,|\delta| \le k$. If $A \subset \Gamma$ has no solution to any of these equations other than $\{a,b\}=\{c,d\}$, then $A$ is called a *$k$-fold Sidon set*. It is easy to see that a $k$-fold Sidon set in a finite abelian group $\Gamma$ has size $O(|\Gamma|^{1/2})$ (see [4] for more details). A 2-fold Sidon set of size roughly $\sqrt{N}/2$ in $\mathbb{Z}_N$ is constructed for infinitely many $N$ in [12], but it is an open question to construct a $k$-fold Sidon set of size $\Theta(\sqrt{N})$ in any abelian group $\Gamma_N$ of order $N$ for any $k \ge 3$. In fact the following is conjectured in [12]:

**Conjecture 2.** *Let $k \in \mathbb{N}$. Then there exists $c_k > 0$ such that for any $N \in \mathbb{N}$, there exists a $k$-fold Sidon set $A \subset [N]$ of size at least $c_k\sqrt{N}$.*

The densest current construction available is due to Ruzsa [15], who showed that for each $k$ there exists a $k$-fold Sidon set of size $N^{1/2-o(1)}$ in $\mathbb{Z}_N$. The construction used for Theorem 1.1 is easily adapted to give counterexamples to Conjecture 1 if there exists a $k$-fold Sidon set of size $\Theta(\sqrt{N})$ in an abelian group $\Gamma_N$ of order $N$. The following theorem is proved in the same way as Theorem 1.1, by taking $G_N = G_{\Gamma_N,\Lambda,S}(C)$ where $C$ is a quadrilateral, $\Lambda=\{0,3,1,2\}$ and $S$ is a 3-fold Sidon set:

**Theorem 7.1.** *If there exists a 3-fold Sidon set of size $\Theta(\sqrt{N})$ in an abelian group $\Gamma$ of order $N$, then there exists a $4N$-vertex graph $G_N$ with $\Theta(N^{3/2})$ edges such that every edge of $G_N$ is contained in exactly one quadrilateral.*

A particularly case is to find a Sidon set of size $\Theta(\sqrt{N})$ in $\mathbb{Z}_N$ such that no difference of two distinct elements equals twice or three times the difference of any other two elements; i.e. the equations $a-b=2(c-d)$ and $a-b=3(c-d)$ force $a=b$ and $c=d$.

• A more general setting is given in [12] in the language of hypergraphs [3]. An $r$-uniform hypergraph has *girth five* if whenever one pair of vertices is selected from each hyperedge, the resulting graph has no cycles of length at most four (and in particular no double edges). In [12] it is conjectured that for every $r \ge 2$, there exists an $r$-uniform hypergraph on $n$ vertices with girth five and $\Theta(n^{3/2})$ hyperedges. This is settled for $r=2$ by Erdős and Rényi [7], and for $r=3$ in [12]. In the case $r=4$, if this conjecture is true then we may place a quadrilateral in each hyperedge to obtain a graph with $\Theta(n^{3/2})$ edges and $n$ vertices, in which every edge is in exactly one quadrilateral. Using Ruzsa's construction [15], one finds for each $r \ge 3$ an $n$-vertex $r$-uniform hypergraph of girth five with $n^{3/2-o(1)}$ edges. In particular, if $f_r(n)$ is the maximum number of hyperedges in an $r$-uniform $n$-vertex hypergraph of girth five, then $f_2(n)=\Theta(n^{3/2})$, $f_3(n)=\frac{1}{6}n^{3/2}+o(n^{3/2})$, and for some constant $c_r>0$,

$$\frac{n^{3/2}}{\exp(c_r\sqrt{\log n})} \le f_r(n) \le \frac{1}{r(r-1)}n^{3/2}+O(n)$$

for all $r \geq 4$.

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