QUALIFYING EXAMS

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1. Summer 2001

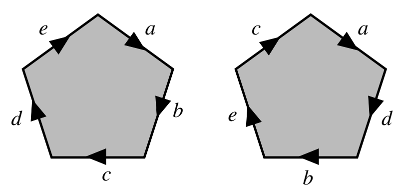
1. Let $p(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_0$ (where $a_n \neq 0$, and $n \geq 1$) be a complex polynomial. Prove by a topological argument that p must have a root in the complex plane.

2. Let Σ_g be a closed orientable surface of genus g. A map $\pi : \Sigma_g \to S^2$ is a *double branched cover* if there is a set $Q = \{p_1, p_2, \ldots, p_n\} \subseteq S^2$ of *branch points*, so that π restricted to $\Sigma_g - \pi^{-1}(Q)$ is a double cover of $S^2 - Q$, but the points p_i have only one preimage each. Use Euler characteristic to find a formula relating g and n.

3. The *connect-sum* (#) of two oriented 4-manifolds is defined by removing an open 4-ball from each, and gluing the resulting manifolds using a homeomorphism between their boundary 3-spheres, in such a way that the orientations match to make a new oriented manifold. Compute the cohomology ring of the connect-sum $X = \mathbb{C}P^2 \# (S^2 \times S^2)$.

4. Let X be a (path-connected) simply-connected CW-complex with $H_2(X) \cong \mathbb{Z} \oplus \mathbb{Z}$ and $H_{\geq 3}(X) = 0$. Prove that X is homotopy-equivalent to the "bouquet of two spheres" $S^2 \vee S^2$.

5. Let X be the result of gluing up the edges of two solid pentagons in pairs, according to the picture shown below. Compute the fundamental group and the homology groups of X. Is it a manifold?



6. Show that any homotopy equivalence from $\mathbb{C}P^{2n}$ to itself is orientation-preserving, i.e. has degree $+1$. Is this true for $\mathbb{C}P^{2n+1}$?

2. Fall 2002

1. Let $E = (\mathbb{R} \times \mathbb{Z}) \cup (\mathbb{Z} \times \mathbb{R})$ be the subset of the plane whose points have at least one of the coordinates an integer. Let $S^1 \vee S^1 \subseteq \mathbb{R}^2 \times \mathbb{R}^2$ be the one-point union of circles. Define $p : E \to S^1 \vee S^1$ by $p(x, y) = (e^{2\pi ix}, e^{2\pi iy})$.

(a). Verify that p is a covering space map.

(b). Let $\sigma : (I, 0) \to (E, (0, 0))$ be the loop which traverses the unit square $I \times \{0, 1\} \cup \{0, 1\} \times I$ once counterclockwise. Prove that $p\sigma$ is the commutator of the two loops of the figure-eight.

(c). Prove that $p_\# : \pi_1(E, (0, 0)) \to \pi_1(S^1 \vee S^1, (1, 1))$ is a monomorphism. Show that this implies that $\pi_1(S^1 \vee S^1, (1, 1))$ is not abelian.

2. Let $T = S^1 \times S^1$ and let Y be its subspace $(S^1 \times \{+1\}) \cup (\{+1\} \times S^1)$, with inclusion map i.

(a). Compute $H_*(T, S^1 \vee S^1; \mathbb{Z})$ and the map $i_* : H_*(S^1 \vee S^1; \mathbb{Z}) \to H_*(T; \mathbb{Z})$, where $i : S^1 \vee S^1 \hookrightarrow T$ is the inclusion map.

(b). Let $Z = S^1 \vee S^1 \vee S^2$ be the one-point union of the two circles and a 2-sphere. Prove that $H_*(Z; \mathbb{Z})$ and $H_*(T; \mathbb{Z})$ are isomorphic, but that Z and T do not have the same homotopy type.

3. (a). Construct a space Y with the following properties:

$$H_k(Y; \mathbb{Z}) = \begin{cases} \mathbb{Z}_4 & \text{if } k = 2 \\ \mathbb{Z} & \text{if } k = 0 \\ 0 & \text{otherwise.} \end{cases}$$

(b). Compute $H_*(\mathbb{R}P^2 \times Y; \mathbb{Z}_2)$, $H_*(\mathbb{R}P^2 \times Y; \mathbb{Z})$, and $H^*(\mathbb{R}P^2 \times Y; \mathbb{Z})$.

4. Let X be a finite-dimensional cell complex with only even-dimensional cells. Prove that $H_*(X; \mathbb{Z})$ is torsion-free.

5. Prove that if $n \geq 1$, then any continuous map $f : \mathbb{R}P^{2n} \to \mathbb{R}P^{2n}$ has a fixed point.

6. Let X be an n-dimensional $\mathbb{Z}_3$-orientable manifold. Prove that X is orientable.

7. Describe submanifold representatives of the generators of the homology groups of $\mathbb{C}P^n$, and explain how to use these to determine the cohomology ring structure.

8. Suppose K is a knot (a smoothly-embedded image of the circle S^1) in S^4. Use transversality to compute the fundamental group of the complement $S^4 - K$.

9. Which of the following functions is a Morse function (having isolated, non-degenerate critical points) on the standard unit sphere $S^2 \subseteq \mathbb{R}^3$?

$$f(x, y, z) = z^2 \qquad f(x, y, z) = z \qquad f(x, y, z) = z^4$$

10. Use transversality to prove that there is no smooth retraction $r : B^n \to S^{n-1}$, and consequently (the Brouwer fixed point theorem) that any smooth automorphism of B^n has a fixed point.

3. Summer 2003

1. Let S, T and K denote the 2-sphere S^2, the 2-torus $S^1 \times S^1$, and the Klein bottle, respectively. For each of the six possibilities $S \to T, T \to S$ etc., either describe a covering map having this form or explain why such a map cannot exist.

2. Let $X = S^1 \vee S^1$ be the figure-of-eight space, and let F_n denote the free group on n generators. By considering covering spaces of X, show that F_2 contains subgroups isomorphic to F_n, for arbitrary $n \geq 3$.

3. Construct a space X having $H_0(X;\mathbb{Z}) \cong \mathbb{Z}$ and $H_5(X;\mathbb{Z}) \cong \mathbb{Z}_4$, but having all other homology groups trivial.

4. Let N be a knotted solid torus in S^3, let T be its boundary torus, and let X be its exterior, that is the closure of $S^3 - N$. Use Mayer-Vietoris to compute the homology $H_*(X;\mathbb{Z})$.

5. Let M^3 be a closed connected oriented 3-manifold with fundamental group isomorphic to the free group on two generators. Compute the homology and cohomology groups $H_*(M;\mathbb{Z})$ and $H^*(M;\mathbb{Z})$.

6. Compute $\mathrm{Ext}(\mathbb{Z} \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_3, \mathbb{Z} \oplus \mathbb{Z}_4 \oplus \mathbb{Z}_5)$.

7. Consider the standard embedding $\mathbb{C}P^1 \subseteq \mathbb{C}P^2$. Show that there is no homeomorphism $f : \mathbb{C}P^2 \to \mathbb{C}P^2$ such that $f(\mathbb{C}P^1)$ is disjoint from $\mathbb{C}P^1$.

8. Describe the universal cover of $X = (S^1 \times S^1) \vee S^2$, and use it to compute the abelian group $\pi_2(X)$.

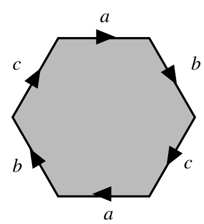
9. Let $E \to S^5$ be a fibre bundle with fibres homeomorphic to S^3. Use the Hurewicz theorem to compute $H_3(E)$.

4. Fall 2003

1. *Commensurability* is the equivalence relation on spaces generated by saying that $X \sim Y$ if X is a finite cover of Y (or vice versa). What are the commensurability classes of closed (not necessarily orientable) 2-dimensional surfaces?

2. Let $X = S^1 \vee S^1$ be the figure-of-eight space. Draw pictures of the covers of X corresponding to the subgroups $\langle abab \rangle$ and $\langle ab, ba \rangle$.

3. Let X be the space obtained by identifying the edges of a solid hexagon as shown below. Compute $H_*(X; \mathbb{Z})$.



4. Let N be submanifold of S^3 which is homeomorphic to a thickened torus $T^2 \times I$. Let X be its exterior, that is the closure of $S^3 - N$. Use Mayer-Vietoris to compute the homology $H_*(X; \mathbb{Z})$.

5. Let M^4 be a closed connected simply-connected 4-manifold. Show that $H_1(M; \mathbb{Z}) = H_3(M; \mathbb{Z}) = 0$ and that $H_2(M; \mathbb{Z})$ is a free abelian group.

6. Compute $\mathrm{Tor}(\mathbb{Z} \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_8, \mathbb{Z} \oplus \mathbb{Z}_4 \oplus \mathbb{Z}_4)$.

7. Consider the standard embedding $\mathbb{C}P^1 \subseteq \mathbb{C}P^2$. Show that any map $f : S^2 \to \mathbb{C}P^2$ whose image $f(S^2)$ is disjoint from $\mathbb{C}P^1$ must be null-homotopic.

8. Describe the universal cover of $X = \mathbb{R}P^3 \vee S^2$, and use it to compute the abelian group $\pi_2(X)$.

5. Fall 2004

1. Find a space X that has the same integral homology and fundamental group as the torus $S^1 \times S^1$, but is not homotopy-equivalent to the torus. Prove that X is not homotopy-equivalent to the torus.

2. Consider the standard covering projection $S^n \to \mathbb{R}P^n$ which maps antipodal points to the same point in $\mathbb{R}P^n$. Prove that the covering projection is not null homotopic.

3. Show that $\mathbb{R}P^k$ is not a retract of $\mathbb{R}P^n$ for $k < n$.

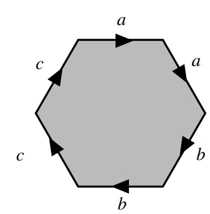
4. Let M be a compact connected nonorientable 3-manifold. Show the first integral homology group of M is infinite.

5. Prove the Borsuk-Ulam theorem that if $n > m \geq 1$, then there is no map $g : S^n \to S^m$ which commutes with the antipodal map.

6. Let M^4 be a closed connected simply-connected 4-manifold. Show that $H_1(M;\mathbb{Z}) = H_3(M;\mathbb{Z}) = 0$ and that $H_2(M;\mathbb{Z})$ is a free abelian group.

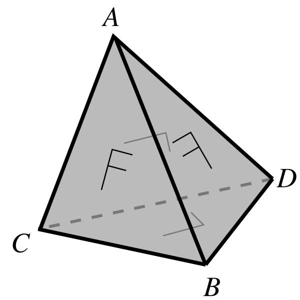
7. Describe the universal cover of $X = \mathbb{R}P^3 \vee S^2$, and use it to compute the abelian group $\pi_2(X)$.

8. Let X be the space obtained by identifying the edges of a solid hexagon as shown below. Compute $H_*(X;\mathbb{Z})$.



6. Summer 2007

1. Consider a solid tetrahedron $ABCD$. The face ABC is glued to ABD by an affine map preserving the order of vertices (i.e. A goes to A, B goes to B, C goes to D.) Similarly, BCD is glued to ACD. Compute the fundamental group of the resulting quotient space.



2. Let X_n be the bouquet of n circles, whose fundamental group (based at the vertex of the bouquet) is the free group F_n on n generators.

(a). Draw a covering of X_3 by X_5. Find the subgroup of $F_3 = \langle a, b, c\rangle$ to which your cover corresponds under the correspondence between subgroups of F_3 and based connected covers of X_3.

(b). Show that X_4 cannot cover X_3.

3. Compute the integral homology $H_*(\mathbb{R}P^2 \times \mathbb{R}P^3; \mathbb{Z})$.

4. Let $T \subseteq S^4$ be a (perhaps knotted) subspace homeomorphic to the 2-torus. Let N be a closed regular neighbourhood of T, so that N is homotopy-equivalent to T. Let X be S^4 minus the interior of N, so that X is a compact 4-manifold with boundary. By considering the relative cohomology $H^*(S^4, N)$ and applying excision and Lefschetz duality, calculate the homology of X.

5. On any closed surface Σ_g of genus $g \geq 1$, it is possible to find a pair of simple closed curves (submanifolds homeomorphic to S^1) meeting transversely once. Use this fact together with intersection theory to show that any map $S^2 \to \Sigma_g$ has degree zero.

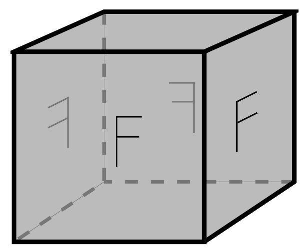
6. Use the Hurewicz theorem to calculate $\pi_3(\mathbb{R}P^3 \vee S^3)$.

7. Show that any closed (i.e. compact, without boundary) 6-manifold which is 2-connected (i.e is path-connected, simply-connected and has $\pi_2 = 0$) must have even Euler characteristic.

8. Let M^3 be a *homology sphere* – a closed 3-manifold having the same homology groups as S^3 – and let $X = \Sigma M$ be its suspension. What are the fundamental group and homology groups of X? Show that X is homotopy-equivalent to S^4.

7. Fall 2007

1. Consider a solid cube. Four of the faces are identified together by means of rigid rotations, as pictured below. Compute the fundamental group of the resulting quotient space.



2. Let Σ_g be the closed orientable surface of genus g, that is the "g-holed torus". Describe all the possible covering spaces of the form $\Sigma_g \to \Sigma_h$, where $1 \leq g, h \leq 4$, and explain why these are the only possibilities.

3. Let X be a space whose homology groups are $\mathbb{Z}, 0, \mathbb{Z}_6$ in dimensions $0, 1, 2$ and zero otherwise. Compute the integral homology $H_*(X \times \mathbb{R}P^3; \mathbb{Z})$.

4. Let K be a (perhaps knotted) subspace of S^5 which is homeomorphic to the 3-sphere. Let N be a closed regular neighbourhood of K, so that N is homotopy-equivalent to K. Let X be S^5 minus the interior of N, so that X is a compact 5-manifold with boundary. By considering the relative cohomology $H^*(S^5, N)$ and applying excision and Lefschetz duality, calculate the homology of X.

5. Let M be a closed (that is, compact and without boundary) path-connected orientable 3-manifold. Suppose that M contains a 2-dimensional orientable submanifold Σ which is *non-separating*, meaning that $M - \Sigma$ is still path-connected. Show that $H_1(M; \mathbb{Z})$ contains a subgroup isomorphic to $\mathbb{Z}$.

6. Use the Hurewicz theorem to calculate $\pi_3(\mathbb{R}P^4 \vee S^3)$.

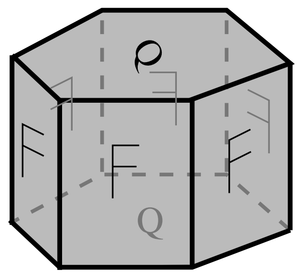
7. Let W be a closed (i.e. compact, without boundary) 4-manifold which is 1-connected (i.e is path-connected and simply-connected). Show that its second homology group is a free abelian group (in other words, has no finite cyclic summands).

8. Show that the Euler characteristic of a closed orientable odd-dimensional manifold is zero. Is this still true if the manifold is non-orientable?

8. Summer 2009

1. Construct a space whose integral homology groups are $\mathbb{Z}, \mathbb{Z}_5, \mathbb{Z}_5, \mathbb{Z}$ in dimensions $0, 1, 2, 3$, and zero otherwise. Does there exist a closed orientable 3-manifold with these homology groups?

2. A space X is constructed by gluing up the solid hexagonal prism shown below: the hexagonal faces are glued using translation and a 60 degree rotation, and the opposite sides of the prism are glued in pairs via translation. Calculate the integral homology of X.

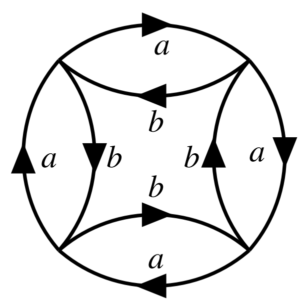


3. Recall that for any $p \geq 2$, the 3-dimensional lens space $L^3(p, 1)$ has integral homology groups $\mathbb{Z}, \mathbb{Z}_p, 0, \mathbb{Z}$ in dimensions $0, 1, 2, 3$. Calculate the integral homology of the product $L(p, 1) \times L(q, 1)$.

4. Let X_n be the bouquet of n circles, whose fundamental group (based at the vertex of the bouquet) is the free group F_n on n generators.

(a). Construct a basepointed covering of X_3 corresponding to the subgroup $\langle b^3, a^2, b^2ab^{-1}\rangle$ of the free group $\langle a, b, c\rangle$.

(b). Find the subgroup of F_2 corresponding to the basepointed cover of X_2 depicted below.



5. Show that if M is a compact orientable manifold with boundary ∂M, then there does not exist a retraction $r : M \to \partial M$.

6. Show that any homotopy equivalence from $\mathbb{C}P^{2n}$ to itself is orientation-preserving, that is has degree $+1$.

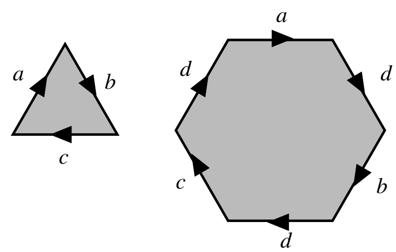
7. Suppose X is a 1-connected CW complex whose homology groups are $\mathbb{Z}$ in dimension 0, $\mathbb{Z}^2$ in dimension 3, and zero otherwise. By constructing a map $S^3 \vee S^3 \to X$, show that X is homotopy-equivalent to $S^3 \vee S^3$.

8. Let M^{2n} be a closed orientable even-dimensional manifold. Show that its Euler characteristic is odd if and only if the dimension of $H_n(M; \mathbb{Q})$ is odd, and that consequently a closed manifold of dimension $4n + 2$ with odd Euler characteristic must be non-orientable.

9. Fall 2009

1. Show that the fundamental group, based at the identity, of a topological group G is abelian.

2. A solid hexagon and a solid triangle are glued together along their edges, according to the following scheme. Calculate the fundamental group and the homology of the resulting space X.



3. Let Y be a space obtained by attaching a 4-ball, via a degree 6 map of its boundary, to a 3-sphere. Calculate the integral homology $H_*(Y \times \mathbb{R}P^2; \mathbb{Z})$.

4. The Euler characteristic $\chi(X)$ of a space X is defined as the alternating sum of the dimensions of the rational homology groups $H_i(X; \mathbb{Q})$. Use Poincaré duality to show that the Euler characteristic of a compact connected closed orientable 3-manifold M^3 is zero. Prove that the result still holds even if M is non-orientable.

5. Show that any homotopy equivalence from $\mathbb{C}P^{2n}$ to itself is orientation-preserving, that is has degree $+1$.

6. Calculate the homotopy group $\pi_3(\mathbb{R}P^4 \vee S^3)$ (where $\vee$ denotes the one-point union of the two spaces).

7. Let M^3 be a *homology sphere*: a connected closed compact 3-manifold with the same homology groups as S^3. Calculate the fundamental group and homology of the suspension ΣM? Use this to show that the suspension is homotopy-equivalent to S^4.

8. Let F_n denote the free group on n generators. Use covering space theory to prove that F_2 contains subgroups isomorphic to F_n, for every $n \geq 1$.

10. Fall 2010

1. Let X be a space formed by gluing two distinct copies of the solid torus $S^1 \times B^2$ along their boundary $S^1 \times S^1$s, via the identity map. Calculate the fundamental group and homology groups of X.

2. How many distinct double covers does the Klein bottle have? Can you identify any of them? (Recall that the Klein bottle can be formed from a square $I \times I$ by identifying opposite edges, one pair in the parallel and one pair in the opposite direction.)

3. Show that a closed, compact, simply-connected 3-manifold M^3 is homotopy-equivalent to S^3.

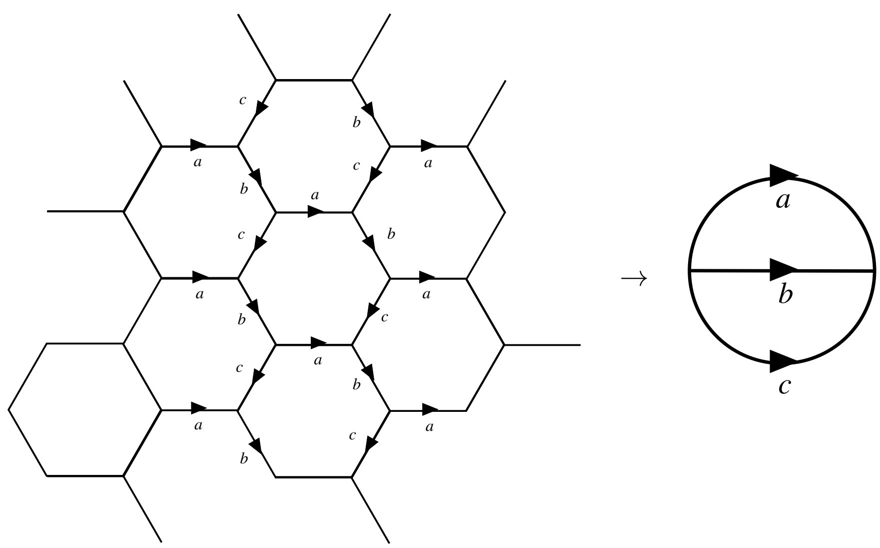
4. Let X be a space whose integral homology groups are $\mathbb{Z}, 0, \mathbb{Z}_8$ in dimensions $0, 1, 2$, and zero otherwise. Compute the integral homology groups of $X \times \mathbb{R}P^3$.

5. Show that there does not exist a map of degree 1 from $S^2 \times S^2$ to $\mathbb{C}P^2$.

6. An orientable closed compact 4-manifold W^4 has a finite fundamental group with d elements, and the rank of its second homology group is r. What is the rank of the second homology group of its universal cover?

7. Use the Hurewicz theorem to calculate π_2 of the space $\mathbb{R}P^2 \vee S^2 \vee S^2$ (that is, the one-point union of a projective plane and two spheres).

8. The infinite hexagonal lattice forms a covering space of the theta graph, as shown below. What is the group of deck translations (covering automorphisms) of the covering?



11. Summer 2011

1. Let M be a simply connected n-dimensional CW complex. Show that any map from M to $\mathbb{R}P^{n+1}$ is homotopic to the constant map.

2. How many distinct double covers does $\mathbb{R}P^3 \times S^1$ have? Can you identify any of them?

3. Let $n \geq 2$ be a positive integer, and let k be in the range $0 < k < n$. Let $X = \mathbb{C}P^n/\mathbb{C}P^k$ be the quotient space obtained from $\mathbb{C}P^n$ by identifying its subspace $\mathbb{C}P^k$ to a point. Calculate the integral cohomology ring of X. (You may assume the cohomology ring of $\mathbb{C}P^n$.)

4. For which n and k (as above) is $X = \mathbb{C}P^n/\mathbb{C}P^k$ homotopy-equivalent to a manifold?

5. For any topological space X, whose total homology is a finitely-generated abelian group, let $\chi(X)$ denote the usual Euler characteristic

$$\chi(X) = \sum (-1)^i \dim_{\mathbb{Q}} H_i(X; \mathbb{Q})$$

and let $\chi_2(X)$ be the "mod-2 homology Euler characteristic"

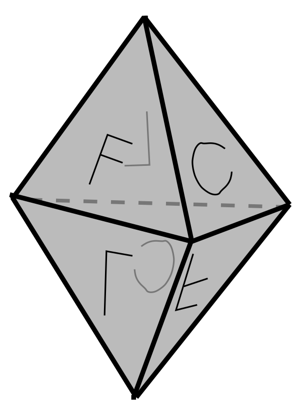
$$\chi_2(X) = \sum (-1)^i \dim_{\mathbb{Z}_2} H_i(X; \mathbb{Z}_2).$$

Use the universal coefficient theorem to show that $\chi(X) = \chi_2(X)$.

6. Let X be a path-connected space with $\pi_{\geq 2}(X) = 0$ and whose fundamental group is a free group on a set S. Show that there is a homotopy equivalence between a bouquet of circles, indexed by S, and X.

7. Show that a closed orientable surface Σ of genus $g \geq 1$ has $\pi_{\geq 2}(\Sigma_g) = 0$, and deduce that the fundamental group of Σ_g is not a free group.

8. Let L be a solid 3-dimensional lens (a flattened ball). Identify the top and bottom surfaces via vertical translation and a twist of 120 degrees, as shown in the picture (where for convenience the lens is drawn as a solid double tetrahedron). Calculate the integral homology of the resulting space.



12. Fall 2011

1. Let $f, g : X \to S^2$ be continuous maps such that for all x in X, $f(x)$ is not antipodal to $g(x)$. Show that f is homotopic to g.

2. Consider the space X obtained from the cylinder $S^1 \times I$ by identifying antipodal points of the circle $S^1 \times \{0\}$, and similarly identifying antipodal points of $S^1 \times \{1\}$. Calculate the fundamental group of X.

3. Assume that X is a path-connected, locally simply-connected space with fundamental group isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3$. How many path-connected covering spaces of X are there, up to equivalence?

4. Consider the set L of 3-manifolds which can be formed by gluing together the boundaries of two solid tori $S^1 \times B^2$ using a homeomorphism. Consider the function $d : L \to \mathbb{N}$ given by the total dimension of its mod-5 homology: $d(M) = \sum \dim H_i(M; \mathbb{Z}_5)$. What is the maximal value of d?

5. Let M be a closed oriented 4-manifold whose second homology $H_2(M; \mathbb{Z})$ has rank 1. Show that there does not exist a free action of the group $\mathbb{Z}_2$ on M.

6. Let P be the Poincaré homology sphere, a 3-manifold whose fundamental group has order 120 and whose universal cover is S^3. Compute π_3 of the one-point union $P \vee S^3$.

7. Let $L(p)$ be a space whose integral homology groups are $\mathbb{Z}, \mathbb{Z}_p, 0, \mathbb{Z}$ in dimensions $0, 1, 2, 3$, and zero otherwise. Let Σ denote the suspension of a space. Compute the cohomology $H^*(\Sigma L(p) \times \Sigma L(q))$.

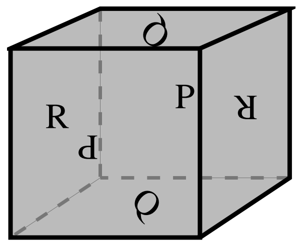
8. Show that there is no self-map of $\mathbb{C}P^2 \times \mathbb{C}P^2$ having degree -1.

13. Summer 2012

1. Let F_n be the free group of rank n, and let H be a subgroup of F_n with index d. Show that H is free, and find its rank.

2. Let X be the space obtained by gluing the two ends of $S^2 \times I$ via the antipodal map of S^2. Compute its homology $H_*(X;\mathbb{Z})$.

3. Let X be the space obtained by gluing opposite pairs of faces of a standard cube I^3 via 180 degree rotations, as shown. Compute the homology $H_*(X;\mathbb{Z})$.



4. Let X be a path-connected space whose homology groups in positive dimensions are $H_k(X;\mathbb{Z}) = \mathbb{Z}/k\mathbb{Z}$. Compute the integer homology $H_*(\mathbb{R}P^2 \times X;\mathbb{Z})$.

5. Compute the first, second and third homotopy groups of $X = \mathbb{R}P^2 \times S^1 \times S^1$.

6. Show that there exists a degree 1 map from $T^3 = S^1 \times S^1 \times S^1$ to S^3, but not vice versa.

7. Show that the second homology group $H_2(X;\mathbb{Z})$ of a closed, path-connected, simply-connected 4-manifold X is free and has rank $\chi(X) - 2$, where $\chi(X)$ is the Euler characteristic.

8. Let Σ be a closed orientable surface of genus 2, whose fundamental group π (with respect to some basepoint x_0) can be described by the presentation

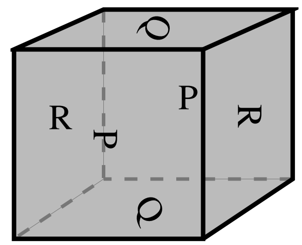
$$\pi = \langle a, b, c, d : aba^{-1}b^{-1}cdc^{-1}d^{-1} = 1\rangle$$

Explain how to associate, to any oriented loop on the surface (not necessarily passing through the basepoint) a conjugacy class in π. Consider two oriented loops α and β on the surface, whose conjugacy classes are represented by the elements $ab^2cda^{-1}b^{-1}c^{-1}d^{-1}$ and $a^2bcda^{-1}b^{-1}c^{-1}d^{-1}$ respectively. Explain why it is impossible to use homotopies of the loops to make α and β disjoint.

14. Fall 2012

1. Let X_n be the bouquet of n circles, whose fundamental group (based at the vertex of the bouquet) is the free group F_n on n generators. Show that X_4 cannot cover X_3, but that X_5 can.

2. Let X be the space obtained by gluing opposite pairs of faces of a standard cube I^3 via 90 degree rotations, as shown. Compute the homology $H_*(X;\mathbb{Z})$.



3. Let Y be a space whose homology groups vanish except for $H_0(Y;\mathbb{Z}) = \mathbb{Z}$ and $H_2(Y;\mathbb{Z}) = \mathbb{Z}_4$. Compute the homology $H_*(\mathbb{R}P^2 \times Y;\mathbb{Z})$ and cohomology $H^*(\mathbb{R}P^2 \times Y;\mathbb{Z})$.

4. Let N be a knotted solid torus in S^3, let T be its boundary torus, and let X be its exterior (that is, the closure of $S^3 - N$). Use Mayer-Vietoris to compute the homology $H_*(X;\mathbb{Z})$.

5. Show that if M is a compact orientable manifold with boundary ∂M, then there does not exist a retraction $r : M \to \partial M$.

6. Let M^4 be a closed connected simply-connected 4-manifold. Show that $H_1(M;\mathbb{Z}) = H_3(M;\mathbb{Z}) = 0$ and that $H_2(M;\mathbb{Z})$ is a free abelian group.

7. Prove the Borsuk-Ulam theorem: that if $n > m \geq 1$, then there is no map $g : S^n \to S^m$ which satisfies $g(-x) = -g(x)$ for all x.

8. Let L be a space which is p-fold covered by S^3, for some $p \geq 1$. Compute the second homotopy group of the one-point union $\pi_2(L \vee S^2)$.

15. Summer 2014

1. Compute the fundamental group and homology groups of the space obtained by removing the union of the three coordinate axes from $\mathbb{R}^3$.

2. Let $K \subseteq V$ be a knotted solid torus $S^1 \times B^2$ inside a larger solid torus $V = S^1 \times B^2$, and let $X = V - \mathring{K}$ be the complement, obtained by removing the interior of K; the picture below shows an example. Compute $H_*(X;\mathbb{Z})$.

3. Let $X = K \times K$ be the product of the Klein bottle K with itself. Compute the homology $H_*(X;\mathbb{Z})$ and cohomology $H^*(X;\mathbb{Z})$.

4. Suppose M is a compact connected 4-manifold without boundary, and that $\pi_1(M) = 1$. Prove that $H_2(M)$ is torsion-free.

5. Show that there is no compact 4-manifold, with or without boundary, which is homotopy-equivalent to $S^2 \vee S^4$.

6. Prove that $\mathbb{R}P^2 \vee S^3$ and $\mathbb{R}P^3$ are not homotopy-equivalent.

7. Suppose $f : M \to N$ is a map between two closed connected oriented n-manifolds which induces an isomorphism $H_*(M) \cong H_*(N)$ (that is, it is a map of degree ± 1). Prove that the induced map $\pi_1(M) \to \pi_1(N)$ must be surjective.

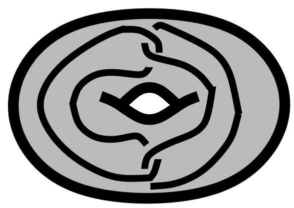
8. Let X be the CW complex formed by attaching k two-cells $e^2_1, \dots e^2_k$ to the circle S^1 $(= e^0 \cup e^1)$ via attaching maps with degrees $n_1, n_2, \dots, n_k$. Compute $\pi_2(X)$ in terms of $n_1, \dots, n_k$.

16. Fall 2014

Three-hour exam. Answer all questions; each is worth the same. You can use standard theorems, but should say when you are doing so. Please try to write good clear mathematics.

1. Compute the fundamental group $\pi_1(X)$ and the homology groups $H_*(X;\mathbb{Z})$ of the space X obtained by removing the 1-skeleton of a regular tetrahedron (that is, the union of the six closed line segments which are the edges) from $\mathbb{R}^3$.

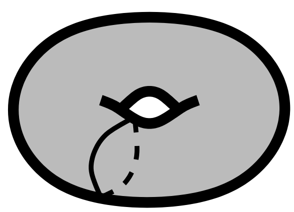
2. Let $L \subseteq V$ be the disjoint union of two solid tori ($S^1 \times B^2$) lying inside a larger solid torus $V = S^1 \times B^2$ as shown below, and let $X = V - \mathring{L}$ be the complement, obtained by removing the interior of L. Compute $H_*(X;\mathbb{Z})$.



3. Let L be a 3-manifold whose homology groups are $\mathbb{Z}, \mathbb{Z}_3, 0, \mathbb{Z}$ in dimensions $0, 1, 2, 3$. Compute the homology $H_*(X;\mathbb{Z})$ and cohomology $H^*(X;\mathbb{Z})$ of the space $X = L \times \Sigma L$. (Here Σ denotes the suspension of a space).

4. Let M be a connected closed oriented 4-manifold whose second homology $H_2(M;\mathbb{Z})$ has rank 1. Show that there does not exist a free action of the group $\mathbb{Z}_2$ on M.

5. Let X be the space obtained by gluing the boundary of a disc to the curve in the torus shown below. Compute the second homotopy group $\pi_2(X)$.



6. Prove that $\mathbb{C}P^2 \# \mathbb{C}P^2$ and $S^2 \times S^2$ are not homotopy-equivalent. (Recall that the *connect-sum* ($\#$) of two closed oriented connected n-manifolds is defined by removing an open n-ball from each and gluing the resulting manifolds using a homeomorphism between their boundary $(n-1)$-spheres, in such a way that the orientations match to make a new closed oriented connected n-manifold.)

7. Suppose $f : M \to N$ is a map of non-zero degree between two closed connected oriented n-manifolds. Prove that for any field $\mathbb{F}$, the induced map $f^* : H^*(N;\mathbb{F}) \to H^*(M;\mathbb{F})$ is injective.

8. Show that a closed connected orientable surface Σ of genus $g \geq 1$ has $\pi_i(\Sigma_g) = 0$ for $i \geq 2$, and deduce that the fundamental group of Σ_g is not a free group.