9. Euler characteristic

1. Suppose C_* is a chain complex whose homology groups H_* are non-zero only in a bounded range of dimensions, and which are themselves abelian groups of finite rank. Then we can define the *Euler characteristic* as the alternating sum

$$\chi = \sum_i (-1)^i \operatorname{rk} H_i.$$

Suppose C_* itself consists of abelian groups of finite rank, non-zero only in a bounded range; then it also makes sense to compute

$$\sum_i (-1)^i \operatorname{rk} C_i.$$

Show that this quantity equals the Euler characteristic!

2. Prove that the Euler characteristic (here thought of as defined on the category of finite CW-complexes) behaves like a "measure" in the following senses:

(a). Let X and Y be finite CW-complexes. Then $\chi(X \times Y) = \chi(X).\chi(Y)$.

(b). Let Z be a finite CW-complex expressible as $Z = X \cup Y$, where X, Y and $X \cap Y$ are all CW-subcomplexes of Z. Then $\chi(Z) = \chi(X) + \chi(Y) - \chi(X \cap Y)$.

(c). Suppose $Y \to X$ is a d-sheeted covering of a finite CW-complex X. Then $\chi(Y) = d.\chi(X)$.

3. (a). Show that the orientable surface Σ_{10} of genus 10 cannot be a covering space of Σ_5.

(b). Give examples to show that the ranks of individual homology groups do not satisfy a simple law like that of Q. 2 (c), and in fact may decrease as well as increase when we go up from X to Y.

4. Prove that the additivity formula of question 2 (b) holds for any topological space Z expressed as a union of open subsets X, Y, as long as the Euler characteristic makes sense for each space (namely that it has finite-rank homology groups, bounded in degree). Hint: Mayer-Vietoris!

5. The orientable surface Σ_g of genus g can be made by attaching a single 2-cell to a bouquet of $2g$ circles along the loop which corresponds to $a_1 b_1 a_1^{-1} b_1^{-1} a_2 b_2 a_2^{-1} b_2^{-1} \cdots a_g b_g a_g^{-1} b_g^{-1}$ (if we identify the fundamental group of the bouquet with the free group on $2g$ generators $a_1, \ldots, a_g, b_1, \ldots, b_g$). What are the (integral) homology groups of Σ_g? Suppose $\Sigma_h \to \Sigma_g$ is a d-sheeted covering, for some positive integer d. Find h in terms of g and d.

6. For any topological space X, whose total homology is a finitely-generated abelian group, let $\chi(X)$ denote the usual Euler characteristic

$$\chi(X) = \sum (-1)^i \dim_{\mathbb{Q}} H_i(X; \mathbb{Q})$$

and let $\chi_2(X)$ be the "mod-2 homology Euler characteristic"

$$\chi_2(X) = \sum (-1)^i \dim_{\mathbb{Z}_2} H_i(X; \mathbb{Z}_2).$$

Use the universal coefficient theorem to show that $\chi(X) = \chi_2(X)$.

7. Write down the Euler characteristics of the following spaces.

(a). $\mathbb{R}P^2$.

(b). $\mathbb{R}P^3$.

(c). $S^2 \vee S^2$.

(d). $S^2 \times S^2$.

(e). $S^2 \times S^3$.

(f). $\mathbb{C}P^3$.

(g). The torus with two discs removed.

(h). The closed orientable surface Σ_4 of genus 4.

(i). $\mathbb{R}^3$ minus the unit circle in the (x, y) plane.

(j). $\mathbb{R}^2$ minus five distinct points.

8. A *triangulation* of a topological space X is a simplicial complex K together with a homeomorphism $h : X \to |K|$. Prove that there are no triangulations of the torus with fewer than fourteen 2-simplices, and exhibit one with fourteen. The Euler characteristic of the torus might be helpful!