6. Homology I

Simplicial homology

1. For each of the following spaces, describe a simplicial (or Δ-) complex representing it, compute its homology groups, and describe their generators: The figure "8", S^2, $\mathbb{R}P^2$, the sphere with an equatorial disc glued to it, the torus with a disc attached inside it.

2. Let K be a simplicial complex with a fixed base vertex a_0, and let $E(K, a_0)$ be the *edge-group* defined as follows. Consider the set of *edge-loops* i.e. strings $a_0a_1 \dots a_n$, where each a_i is a vertex, $a_n = a_0$, and each (a_i, a_{i+1}) is a (0- or 1-) simplex. Define the obvious composition, and work modulo the two relations of *edge-homotopy*, which say that

$$a_0a_1 \dots a_{i-1}a_ia_{i+1} \dots a_n \sim a_0a_1 \dots a_{i-1}a_{i+1} \dots a_n$$

provided (a_{i-1}, a_i, a_{i+1}) spans a (0-, 1- or 2-) simplex, and that a_{i+1} may be cancelled from the string if it equals a_i. Show that there is a surjection $E(K, a_0) \to H_1(K)$ with kernel the commutator subgroup, and prove $E(K, a_0) \cong \pi_1(|K|, a_0)$ by using van Kampen's theorem (it will probably help to take a maximal tree).

3. Let K be a simplicial complex with no simplexes of dimension higher than n. Suppose that every $(n-1)$-simplex is a face of exactly two n-simplexes, and that any two n-simplexes may be connected by a finite sequence of n-simplexes, each adjacent by an $(n-1)$-face to the last. Show that (simplicial) $H_n(K)$ is either trivial or isomorphic to $\mathbb{Z}$; in the latter case a generator being the sum of all the n-simplexes (suitably oriented).

Singular homology

4. Let X be a path-connected space with basepoint x_0.

(a) Show that there is a natural map $h : \pi_1(X, x_0) \to H_1(X)$ which is a homomorphism.

(b) Show that it factors through the abelianisation of $\pi_1(X, x_0)$; that is, the group obtained by quotienting by the subgroup generated by all commutators $aba^{-1}b^{-1}$.

(c) Show that h is surjective. (Hint: given a 1-cycle z, look at the set of endpoints of its singular 1-simplexes, and choose a path joining each one to x_0.)

(d) Show that h is injective. (Hint: if some loop α is the boundary of a 2-chain u, join the corners of the 2-simplexes in u to the basepoint in a similar way, so as to write α as a large composite of loops. Then try to show that by reordering these (working modulo the commutator subgroup) it can be made null-homotopic.)

5. Let a singular homology class $c \in H_n(X)$ be represented by a singular n-cycle z. Use the idea of question 3 to construct a simplicial complex K, a homology class $\mu \in H_n(K)$, and a map $f : |K| \to X$ such that $f_*(\mu) = c$. Show that $|K|$, while not necessarily an n-manifold, is an *n-circuit*, i.e. that its set of singular points (ones without a neighbourhood homeomorphic to $\mathbb{R}^n$) is a subcomplex of dimension less than or equal to $n-2$. This gives rise to a useful and very visual representation of homology classes by *manifolds with singularities of codimension* ≥ 2.

Some algebra

6. Let $0 \to A \xrightarrow{i} B \xrightarrow{p} C \to 0$ be a short exact sequence of abelian groups. Show that the following conditions are equivalent:

(a). There exists a *retraction*: a homomorphism $r : B \to A$ with $r \circ i = 1_A$

(b). There exists a *section*: a homomorphism $s : C \to B$ with $p \circ s = 1_C$

(c). The sequence *splits*: there exists an isomorphism θ between B and $A \oplus C$ so that

$$\begin{CD}
0 @>>> A @>i>> B @>p>> C @>>> 0\\
@. @VV\mathrm{id}_AV @VV\theta V @VV\mathrm{id}_CV @.\\
0 @>>> A @>>> A \oplus C @>>> C @>>> 0
\end{CD}$$

commutes (where on the bottom row the maps are the obvious inclusion and projection). Show that if C is free, such a sequence always splits.

7. Show that a long exact sequence may be written as a collection of short exact sequences. For each exact sequence of abelian groups below, say as much as possible about the unknown group G and homomorphism α:

(a). $0 \to \mathbb{Z}_2 \to G \to \mathbb{Z} \to 0$

(b). $0 \to \mathbb{Z} \to G \to \mathbb{Z}_2 \to 0$

(c). $0 \to \mathbb{Z} \xrightarrow{\alpha} \mathbb{Z} \oplus \mathbb{Z} \to \mathbb{Z} \oplus \mathbb{Z}_2 \to 0$

(d). $0 \to G \to \mathbb{Z} \xrightarrow{\alpha} \mathbb{Z} \to \mathbb{Z}_2 \to 0$

(e). $0 \to \mathbb{Z}_3 \to G \to \mathbb{Z}_2 \to \mathbb{Z} \xrightarrow{\alpha} \mathbb{Z} \to 0$

8. *The 5-lemma*: show that if the rows are exact, and the outer four 'f's are isomorphisms, then the middle one is also. Can you relax the hypotheses a little and still get the conclusion?

$$\begin{CD}
A_1 @>\alpha_1>> A_2 @>\alpha_2>> A_3 @>\alpha_3>> A_4 @>\alpha_4>> A_5\\
@VVf_1V @VVf_2V @VVf_3V @VVf_4V @VVf_5V\\
B_1 @>>\beta_1> B_2 @>>\beta_2> B_3 @>>\beta_3> B_4 @>>\beta_4> B_5
\end{CD}$$