PROBLEM SHEETS

6. Homology I

Simplicial homology

1. For each of the following spaces, describe a simplicial (or Δ -) complex representing it, compute its homology groups, and describe their generators: The figure "8", S^2 , $\mathbb{R}P^2$, the sphere with an equatorial disc glued to it, the torus with a disc attached inside it.

2. Let K be a simplicial complex with a fixed base vertex a_0 , and let $E(K, a_0)$ be the *edge-group* defined as follows. Consider the set of *edge-loops* i.e. strings $a_0a_1 \ldots a_n$, where each a_i is a vertex, $a_n = a_0$, and each (a_i, a_{i+1}) is a (0- or 1-) simplex. Define the obvious composition, and work modulo the two relations of *edge-homotopy*, which say that

$$a_0a_1\ldots a_{i-1}a_ia_{i+1}\ldots a_n \sim a_0a_1\ldots a_{i-1}a_{i+1}\ldots a_n$$

provided (a_{i-1}, a_i, a_{i+1}) spans a (0, 1- or 2-) simplex, and that a_{i+1} may be cancelled from the string if it equals a_i . Show that there is a surjection $E(K, a_0) \to H_1(K)$ with kernel the commutator subgroup, and prove $E(K, a_0) \cong \pi_1(|K|, a_0)$ by using van Kampen's theorem (it will probably help to take a maximal tree).

3. Let K be a simplicial complex with no simplexes of dimension higher than n. Suppose that every (n-1)-simplex is a face of exactly two n-simplexes, and that any two n-simplexes may be connected by a finite sequence of n-simplexes, each adjacent by an (n-1)-face to the last. Show that (simplicial) $H_n(K)$ is either trivial or isomorphic to \mathbb{Z} ; in the latter case a generator being the sum of all the n-simplexes (suitably oriented).

Singular homology

4. Let X be a path-connected space with basepoint x_0 .

(a) Show that there is a natural map $h: \pi_1(X, x_0) \to H_1(X)$ which is a homomorphism.

(b) Show that it factors through the abelianisation of $\pi_1(X, x_0)$; that is, the group obtained by quotienting by the subgroup generated by all commutators $aba^{-1}b^{-1}$.

(c) Show that h is surjective. (Hint: given a 1-cycle z, look at the set of endpoints of its singular 1-simplexes, and choose a path joining each one to x_0 .)

(d) Show that h is injective. (Hint: if some loop α is the boundary of a 2-chain u, join the corners of the 2-simplexes in u to the basepoint in a similar way, so as to write α as a large composite of loops. Then try to show that by reordering these (working modulo the commutator subgroup) it can be made null-homotopic.)

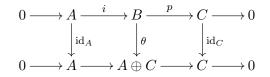
5. Let a singular homology class $c \in H_n(X)$ be represented by a singular *n*-cycle *z*. Use the idea of question 3 to construct a simplicial complex *K*, a homology class $\mu \in H_n(K)$, and a map $f : |K| \to X$ such that $f_*(\mu) = c$. Show that |K|, while not necessarily an *n*-manifold, is an *n*-circuit, i.e. that its set of singular points (ones without a neighbourhood homeomorphic to \mathbb{R}^n) is a subcomplex of dimension less than or equal to n-2. This gives rise to a useful and very visual representation of homology classes by manifolds with singularities of codimension ≥ 2 .

Some algebra

6. Let $0 \to A \xrightarrow{i} B \xrightarrow{p} C \to 0$ be a short exact sequence of abelian groups. Show that the following conditions are equivalent:

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- (a). There exists a *retraction*: a homomorphism $r: B \to A$ with $r \circ i = 1_A$
- (b). There exists a section: a homomorphism $s: C \to B$ with $p \circ s = 1_C$
- (c). The sequence *splits*: there exists an isomorphism θ between B and $A \oplus C$ so that



commutes (where on the bottom row the maps are the obvious inclusion and projection). Show that if C is free, such a sequence always splits.

7. Show that a long exact sequence may be written as a collection of short exact sequences. For each exact sequence of abelian groups below, say as much as possible about the unknown group G and homomorphism α :

(a). $0 \to \mathbb{Z}_2 \to G \to \mathbb{Z} \to 0$ (b). $0 \to \mathbb{Z} \to G \to \mathbb{Z}_2 \to 0$ (c). $0 \to \mathbb{Z} \xrightarrow{\alpha} \mathbb{Z} \oplus \mathbb{Z} \to \mathbb{Z} \oplus \mathbb{Z}_2 \to 0$ (d). $0 \to G \to \mathbb{Z} \xrightarrow{\alpha} \mathbb{Z} \to \mathbb{Z}_2 \to 0$ (e). $0 \to \mathbb{Z}_3 \to G \to \mathbb{Z}_2 \to \mathbb{Z} \xrightarrow{\alpha} \mathbb{Z} \to 0$

8. The 5-lemma: show that if the rows are exact, and the outer four 'f's are isomorphisms, then the middle one is also. Can you relax the hypotheses a little and still get the conclusion?

$$A_{1} \xrightarrow{\alpha_{1}} A_{2} \xrightarrow{\alpha_{2}} A_{3} \xrightarrow{\alpha_{3}} A_{4} \xrightarrow{\alpha_{4}} A_{5}$$

$$\downarrow f_{1} \qquad \downarrow f_{2} \qquad \downarrow f_{3} \qquad \downarrow f_{4} \qquad \downarrow f_{5}$$

$$B_{1} \xrightarrow{\beta_{1}} B_{2} \xrightarrow{\beta_{2}} B_{3} \xrightarrow{\beta_{3}} B_{4} \xrightarrow{\beta_{4}} B_{5}$$