14. Homotopy theory

Constructions

1. Define the *smash product* of based spaces $X \wedge Y$ as the quotient space $(X \times Y)/(X \vee Y)$, where $X \vee Y$ is the subspace $(X \times \{y_0\}) \cup (\{x_0\} \times Y)$. Prove that this is the correct definition of coproduct in the category of based spaces. Show that $S^1 \wedge X$ is another way of defining the reduced suspension ΣX of a based space, and that $S^n \wedge S^m \cong S^{m+n}$.

2. Use the homotopy extension principle to prove that the cone CK on a CW-complex K is contractible.

3. Show that the homotopy extension principle for (X, A) is satisfied if and only if there is a retraction (it doesn't have to be a deformation retraction) $r : X \times I \to X \times \{0\} \cup A \times I$.

Homotopy groups

4. Suppose that $x_0 \in A \subseteq X$, and that there is a retraction $r : X \to A$. Prove that

$$\pi_n(X, x_0) \cong \pi_n(X, A, x_0) \oplus \pi_n(A, x_0).$$

5. For any based space (X, x_0) we can consider the set of *free homotopy* classes $[S^n, X]$, defined without reference to the basepoint, as well as the usual set of *based homotopy* classes $[S^n, X]_0 = \pi_n(X, x_0)$. There is a natural map $[S^n, X]_0 \to [S^n, X]$; show that if X is path-connected, this map identifies $[S^n, X]$ as the set of orbits in $\pi_n(X, x_0)$ under the action of $\pi_1(X, x_0)$.

6. Show that for a path-connected H-space Y, the action of $\pi_1(Y, y_0)$ on $\pi_{n\geq 1}(Y, y_0)$ is trivial.

7. One proof that a topological group G has abelian fundamental group $\pi_1(G, 1)$ goes as follows. Define operations $*$ (path-composition) and $\cdot$ (pointwise multiplication) on the set $\Omega_1 Y$ of loops based at the identity. They satisfy $(a * b) \cdot (c * d) = (a \cdot c) * (b \cdot d)$, and suitable constant loop substitutions give the result. Use this idea to show that

(a) for an H-space Y, $\pi_1(Y)$ is abelian (and the two group operations coincide).

(b) the natural group structures on the homotopy sets $[\Sigma\Sigma X, Y]_0$ and $[X, \Omega\Omega Y]_0$ are abelian, for any (based spaces) X and Y.

Higher homotopy

8. If $f : X \to Y$ is a map, define the *mapping cone* of f to be $CX \cup_f Y$, where CX is the cone on X, and the gluing is the obvious one. Suppose f maps S^{2n-1} to $\mathbb{C}P^{n-1}$; what does the mapping cone look like? Use this to show that $\pi_{2n-1}(\mathbb{C}P^{n-1}) \neq 0$.

9. Give an example of a CW-pair (K, L) such that $\pi_*(K, L) \not\cong \pi_*(K/L)$.

10. Let (X, A) be an arbitrary pair of topological spaces with $\pi_i(X, A) = 0$ for $i \leq n$. Suppose K is a CW-complex of dimension less than n. Show that $[K, X] = [K, A]$. Use this lemma to prove that a weak homotopy equivalence $f : X \to Y$ between arbitrary spaces induces a bijection $f_* : [K, X] \to [K, Y]$, whenever K is a CW-complex.

11. A CW-complex X which has only one non-vanishing homotopy group $\pi_n(X) \cong \pi$ is called an *Eilenberg-MacLane space* $K(\pi, n)$.

(a). Recall how to build, for any group π, a path-connected CW-complex with fundamental group π. Show how to attach cells of dimension ≥ 3 so as to kill all the higher homotopy groups and thereby construct a $K(\pi, 1)$.

(c). Use a similar procedure to prove that for any $n \geq 1$ and group π, there exists a space of type $K(\pi, n)$, provided only that the obvious restriction holds: π must be abelian if $n \geq 2$.

Whitehead

12. Let X be a path-connected CW-complex with $\pi_{\geq 2}(X) = 0$ and $\pi_1(X)$ being a free group on a set S. Show that there is a homotopy equivalence between X and a bouquet of circles indexed by S.

13. These example show the necessity of hypotheses in various forms of Whitehead's theorem.

(a). Show that $S^2 \times \mathbb{R}P^n$ and $\mathbb{R}P^2 \times S^n$ have the same homotopy groups, but (for $n \geq 2$) different homotopy type.

(b). Show that the subspace of the plane $\{x = 0\} \cup \{(x, \sin(1/x)) : x \neq 0\}$ is weakly homotopy-equivalent to, but not homotopy-equivalent to, the three-point space.

(c). Find a pair of 1-connected spaces with the same homology but different homotopy type.

(d). Let $X = (S^2 \vee S^1) \cup B^3$, where the 3-cell attaches via the element $2t - 1 \in \pi_2(S^2 \vee S^1) \cong \mathbb{Z}[t, t^{-1}]$. Show that inclusion of the circle into X induces isomorphisms of fundamental groups and homology but that it is not a homotopy equivalence.

Hurewicz

14. Use the universal cover to calculate the second homotopy groups $\pi_2(S^2 \vee S^1)$ and $\pi_2(S^2 \vee \mathbb{R}P^2)$ and their $\mathbb{Z}\pi_1$-module structures.

15. Let X be a 1-connected CW-complex with $H_2(X) \cong \mathbb{Z} \oplus \mathbb{Z}$ and $H_{\geq 3}(X) = 0$. Prove that X is homotopy-equivalent to the "bouquet of two spheres" $S^2 \vee S^2$.

16. Let $M = S^3 - N(K)$ be a *knot complement*, where K is a ("tamely" i.e. polygonally) embedded circle in the 3-sphere, and $N(K)$ is an open solid torus neighbourhood; thus M is a compact 3-manifold with boundary $S^1 \times S^1$. It can be shown that $\pi_2(M) = 0$; use this to prove that the only non-vanishing homotopy group of M is π_1, so that "M is a $K(\pi, 1)$".