13. Poincaré duality and intersection theory

Orientation and Poincaré duality

1. Use the idea of the orientation homomorphism to show that the Klein bottle cannot be triple-covered by the torus; more generally, a finite cover of a non-orientable manifold by an orientable path-connected orientable must have an even number of sheets.

2. Show that the orientation double cover $\tilde{M} \to M$ of any manifold is orientable, and in fact that it has a *canonical* orientation.

3. Show that a compact manifold cannot retract to its boundary.

4. Recall that for a topological space X with finite-dimensional homology, the pairings

$$H^i(X;\mathbb{Q}) \times H_i(X;\mathbb{Q}) \to \mathbb{Q}$$

are non-degenerate. Use this fact and Poincaré duality to show that if X is an odd-dimensional compact oriented manifold, its Euler characteristic is zero. Show that in fact this holds even for non-orientable manifolds.

5. Show that a closed n-manifold with odd Euler characteristic can never be the boundary of a compact $n+1$-manifold.

6. Let $n \geq 2$ be a positive integer, and let k be in the range $0 < k < n$. Let $X = \mathbb{C}P^n/\mathbb{C}P^k$ be the quotient space obtained from $\mathbb{C}P^n$ by identifying its subspace $\mathbb{C}P^k$ to a point. Calculate the integral cohomology ring of X. (You may assume the cohomology ring of $\mathbb{C}P^n$.) For which n and k (as above) is $X = \mathbb{C}P^n/\mathbb{C}P^k$ homotopy-equivalent to a manifold?

7. Let M^3 be a closed connected oriented 3-manifold with fundamental group isomorphic to the free group on two generators. Compute the homology and cohomology groups $H_*(M;\mathbb{Z})$ and $H^*(M;\mathbb{Z})$.

8. Show that any closed 1-connected (that is, path-connected and simply-connected) 3-manifold is a *homology sphere* – that is, it has the same homology groups as the 3-sphere.

Degree

9. For connected closed oriented manifolds M^n, N^n of the same dimension, the *degree* of a map $f : M \to N$ is defined by the identity $f_*[M] = \deg(f)[N]$ in $H_n(N;\mathbb{Z})$. Use cohomology rings to show that any map from the 2-sphere to the torus has degree zero.

10. Show that every closed oriented n-manifold M^n has a degree 1 map *to* S^n, but that not every one has a degree 1 map *from* S^n.

11. Let M^n be a connected closed oriented manifold, and N a connected d-sheeted cover of M. Check that N is a closed manifold, that it can be oriented in a natural way, and that the degree of the covering map is d.

Intersection theory

12. Compute the cohomology $H^*(S^2 \times S^4)$ using a cellular decomposition. Describe the generators, and use intersection theory to compute the ring structure. Show that $S^2 \times S^4$ is not homotopy-equivalent to $\mathbb{C}P^3$.

13. The *connect-sum* (#) of two oriented n-manifolds is defined by removing an open n-ball from each, and gluing the resulting manifolds using an *orientation-reversing* homeomorphism between their boundary $(n-1)$-spheres; this makes the new manifold canonically oriented. Compute the cohomology ring of the connect-sum $M^4 = \mathbb{C}P^2 \#(S^2 \times S^2)$.

14. Recall the computation of the homology and cohomology of $\mathbb{R}P^n$ with mod-2 coefficients. Describe submanifolds representing the generators of the mod-2 homology groups. Use intersections between these to compute the ring structure on $H^*(\mathbb{R}P^n; \mathbb{Z}_2)$.

15. Consider the standard embedding $\mathbb{C}P^1 \subseteq \mathbb{C}P^2$. Show that there is no homeomorphism $f : \mathbb{C}P^2 \to \mathbb{C}P^2$ such that $f(\mathbb{C}P^1)$ is disjoint from $\mathbb{C}P^1$.

16. Suppose that Σ is a connected closed oriented surface, and γ is a simple closed curve (a subspace homeomorphic to the circle) on Σ. We define γ to be either *separating* or *non-separating* according to whether the surface $\Sigma - \gamma$ is disconnected or connected. Show that γ is separating if and only if its homology class in $H_1(\Sigma; \mathbb{Z})$ is zero.

17. Let $M = \Sigma \times S^1$, where Σ is a (closed orientable) genus-two surface. Compute the homology of M (you can use any method you want) and describe submanifold representatives for generators of the homology groups.

18. Consider the closed solid torus $V = S^1 \times B^2$, with boundary $T = S^1 \times S^1$. Compute, and describe generators for, the following invariants:

(a). The homology groups $H_*(V)$;

(b). The relative homology groups $H_*(V, T)$;

(c). The cohomology groups $H^*(V)$ and $H^*(V, T)$.

Describe the intersection pairings $H_i(V) \times H_{3-i}(V, T) \to \mathbb{Z}$ in terms of your generators.