Marsden and Tromba problem 8.2 # 15 - solutions and comments

The problem.

Let Σ be the surface given by $x^2 + y^2 + z^2 = 1$ and $x + y + z \geq 1$, oriented with the outward normal. Let $\mathbf{F}$ be the vector field given by $\mathbf{F} = (x, y, z) \times (1, 1, 1)$. Compute

$$\int_\Sigma (\nabla \times \mathbf{F}) \cdot \mathrm{d}\mathbf{A}.$$

Methods.

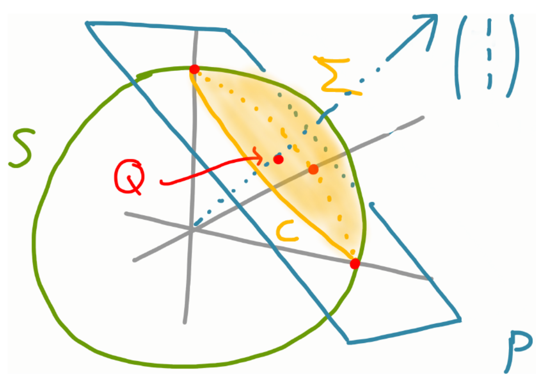
This is a great problem because it can be used to illustrate various different geometric insights and shortcuts. Here are five possible ways to do it.

1. Completely direct: just calculate the flux of $\nabla \times \mathbf{F}$ through the surface
2. Use Stokes theorem directly: calculate the circulation of $\mathbf{F}$ around $\partial\Sigma$
3. Use Stokes indirectly: change the surface to a disc and calculate the flux over this disc
4. Use the special rotational nature of the vector field
5. Use the rotational symmetry of the entire problem.

I think (3) is probably the best and (1) probably the worst, but the difficulty and/or value of the others is more a matter of taste!

Some geometry.

Whichever method we use, we'll need some of the following geometric information:



Let S be the unit sphere $x^2 + y^2 + z^2 = 1$.

Let P be the plane $x + y + z = 1$.

Let L be the line in the direction $(1, 1, 1)$.

Let C be the circle $P \cap S = \partial\Sigma$.

C contains points $(1, 0, 0)$, $(0, 1, 0)$ and $(0, 0, 1)$.

The centre of C is $Q = L \cap P = Q = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$.

The radius of C is the distance from Q to (say) $(1, 0, 0)$, which is $\sqrt{(\frac{2}{3})^2 + (\frac{1}{3})^2 + (\frac{1}{3})^2} = \sqrt{\frac{2}{3}}$.

The distance of Q (and hence P) from the origin is $\sqrt{(\frac{1}{3})^2 + (\frac{1}{3})^2 + (\frac{1}{3})^2} = \frac{1}{\sqrt{3}}$.

Explicit formulae for the vector field and its curl are

$$\mathbf{F} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \times \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} y - z \\ z - x \\ x - y \end{pmatrix} \qquad \nabla \times \mathbf{F} = \begin{pmatrix} -2 \\ -2 \\ -2 \end{pmatrix}.$$

Method 2: direct calculation of the circulation.

Here we begin by using Stokes' theorem, then compute directly the circulation of $\mathbf{F}$ around C. How do we parametrise this circle? One way is to begin with the parametrisation of the standard unit circle $\theta \mapsto \cos\theta.\mathbf{i} + \sin\theta.\mathbf{j}$ in the xy-plane, and then make various adjustments to get what we want. We know we can scale this to make the radius $\sqrt{2/3}$, and we can add $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ to move the centre from the origin to the point Q at the centre of C.

To alter the plane of the circle so that it lies in the plane P, we just need to replace $\mathbf{i}$ and $\mathbf{j}$ by a pair of orthogonal unit vectors lying in P. For example, take the vector from Q to $(1, 0, 0)$ and rescale it to be a unit vector, giving $\mathbf{a} = (2, -1, -1)/\sqrt{6}$. Then let $\mathbf{b} = \hat{\mathbf{n}} \times \mathbf{a}$, where $\hat{\mathbf{n}} = (1, 1, 1)/sqrt3$ is the unit normal to P: this gives us unit vector lying in P and orthogonal to $\mathbf{a}$, namely $\mathbf{b} = (0, 1, -1)/\sqrt{2}$. Summing up, we get a parametrisation

$$\mathbf{s}(\theta) = \begin{pmatrix} \frac{1}{3} \\ \frac{1}{3} \\ \frac{1}{3} \end{pmatrix} + \cos\theta \frac{1}{\sqrt{6}} \begin{pmatrix} 2 \\ -1 \\ -1 \end{pmatrix} + \sin\theta \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}.$$

It's messy but not impossible to compute the circulation using this parametrisation.

Method 1: direct calculation of the flux.

If we wanted to parametrise Σ itself, so as to integrate $\nabla \times \mathbf{F}$ over it directly, we could begin as above, picking an orthonormal basis of vectors $\mathbf{a}, \mathbf{b}, \hat{\mathbf{n}}$ aligned nicely with the plane P. The first two vectors point in the plane of P while the third one is orthogonal to it, so we can set up a new spherical coordinate system aligned with the plane by taking the standard formulae involving θ and ϕ, but replacing the $(\mathbf{i}, \mathbf{j}, \mathbf{k})$ basis with the $(\mathbf{a}, \mathbf{b}, \hat{\mathbf{n}})$ basis, so that

$$(\theta, \phi) \mapsto \cos\theta \sin\phi.\mathbf{a} + \sin\theta \sin\phi.\mathbf{b} + \cos\phi.\hat{\mathbf{n}}$$

To describe the cap Σ we require the limits $0 \leq \theta \leq 2\pi$ and $0 \leq \phi \leq \cos^{-1}(1/\sqrt{3})$, the latter because the boundary of Σ corresponds to the component of $\hat{\mathbf{n}}$ being $1/\sqrt{3}$.

Method 3: change the surface.

Any time we have to integrate a curl $\nabla \times \mathbf{F}$ over a surface Σ, we know by Stokes' theorem that the result depends only on $\mathbf{F}$ and the *boundary* $\partial\Sigma$ of the surface. This shows, for example, that if we wiggle the *interior* of the surface around in space (keeping its boundary fixed), we won't change the integral of $\nabla \times \mathbf{F}$. But more dramatically, we could just *replace* Σ by *any other oriented surface* Σ' *which has the same (oriented) boundary*, and just integrate over Σ' instead!

In our problem, the simplest surface whose boundary is C is not the curved cap Σ, but the flat disc D lying inside C in the plane $x + y + z = 1$, oriented outwards from the origin. So we replace

$$\int_\Sigma (\nabla \times \mathbf{F}) \cdot \mathrm{d}\mathbf{A} = \int_D (\nabla \times \mathbf{F}) \cdot \mathrm{d}\mathbf{A}.$$

Because D lies flat in the plane P, it has a constant unit normal $\hat{\mathbf{n}} = \frac{1}{\sqrt{3}}(1, 1, 1)$, which means

$$\int_D (\nabla \times \mathbf{F}) \cdot \mathrm{d}\mathbf{A} = \int_D ((\nabla \times \mathbf{F}) \cdot \hat{\mathbf{n}})\, \mathrm{d}A = \int_D \begin{pmatrix} -2 \\ -2 \\ -2 \end{pmatrix} \cdot \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}\, \mathrm{d}A = -\frac{6}{\sqrt{3}} \operatorname{area}(D).$$

Since the radius of D is $\sqrt{2/3}$, its area is $2\pi/3$ and we get the answer $-4\pi/\sqrt{3}$.

Method 4: using the special nature of the vector field.

Vector fields of the form $\mathbf{F} = \mathbf{a} \times (x, y, z)$ (where $\mathbf{a}$ is some fixed vector) appear often in physics. That's because such a vector field generates a *rigid rotational flow* anticlockwise about the line A through the origin in the direction $\mathbf{a}$, and the angular velocity of this rotation is given by the magnitude $\|\mathbf{a}\|$: each flow line is a circle (centred on the axis A and lying in a plane orthogonal to A) and each particle takes the same time $2\pi/\|\mathbf{a}\|$ to go right the way around its flow line.

The vector field $\mathbf{F} = (x, y, z) \times (1, 1, 1) = -(1, 1, 1) \times (x, y, z)$ therefore corresponds to a *clockwise* rotational field about our line L. What is nice about this is that the circle C actually is a flow line of $\mathbf{F}$, and therefore $\mathbf{F}$ is everywhere *tangent* to C, and has constant magnitude on C.

This means that if we use Stokes' theorem, we can easily evaluate the circulation integral

$$\int_\Sigma (\nabla \times \mathbf{F}) \cdot \mathrm{d}\mathbf{A} = \int_C \mathbf{F} \cdot \mathrm{d}\mathbf{s} = \int_C (\mathbf{F} \cdot \hat{\mathbf{t}})\ \mathrm{d}s$$

as long as we're careful with signs! We are travelling around C *anticlockwise* (this is the orientation coming from Σ having the outward normal), whereas $\mathbf{F}$ points in the *clockwise* tangent direction. Therefore $\mathbf{F} \cdot \hat{\mathbf{t}} = -\|\mathbf{F}\|$ and the final answer will be *minus* the magnitude of $\mathbf{F}$, times the length of C (which is 2π times its radius $\sqrt{2/3}$).

The magnitude $\|\mathbf{F}\|$ can also be calculated in several ways! We could think of the cross product $(x, y, z) \times (1, 1, 1)$ geometrically: the first vector is a unit vector (as we're on the unit sphere) and the second has length $\sqrt{3}$, and we can compute the angle between them by returning to the picture. Or we could simply work it out at *one* point (x, y, z) – say $(1, 0, 0)$ on C – because we know it's constant on C; that gives $(1, 0, 0) \times (1, 1, 1) = (0, -1, 1)$, so the magnitude is $\sqrt{2}$. Yet another way is to say that if a particle goes around a circle of radius $\sqrt{2/3}$ with angular velocity $\|(1, 1, 1)\| = \sqrt{3}$ then its linear tangential velocity is the product of these numbers, which is $\sqrt{2}$.

In any case, multiplying $-\sqrt{2}$ by the circumference $2\pi\sqrt{2/3}$ gives us, once again, $-4\pi/\sqrt{3}$.

Method 5: rotate everything.

If the slicing plane were horizontal, the problem would be much easier, since we could use standard spherical coordinates to parametrise things. As it stands, computing directly (method 1 or 2) requires us to essentially create and working with a non-standard spherical coordinate system whose axis is aligned with $(1, 1, 1)$ instead of the z-axis. This is of course a bit painful!

There is an alternative approach: instead of rotating our coordinate system to be better adapted to the problem, why don't we rotate the problem to be better adapted to our standard coordinate system? The idea is this: if we choose R to be any kind of rigid rotation about the origin then we can use it to move the domain of integration, as long as we "move" the integrand in the same way.

Remember first that *scalar product is preserved by rotation.* For any vectors $\mathbf{a}, \mathbf{b}$ we have

$$R(\mathbf{a}) \cdot R(\mathbf{b}) = \mathbf{a} \cdot \mathbf{b}.$$

This means that if $\mathbf{G}$ is any vector field and Σ any surface then

$$\int_\Sigma \mathbf{G} \cdot \mathrm{d}\mathbf{A} = \int_\Sigma R(\mathbf{G}) \cdot R(\mathrm{d}\mathbf{A}) = \int_{R(\Sigma)} R(\mathbf{G}) \cdot \mathrm{d}\mathbf{A}.$$

(Think with Riemann sums: the set of '$R(\mathrm{d}\mathbf{A})$'s we run over when $\mathrm{d}\mathbf{A}$ belongs to Σ is the same as the set of '$\mathrm{d}\mathbf{A}$'s we run over when $\mathrm{d}\mathbf{A}$ belongs to $R(\Sigma)$.) We also need two additional facts of the same nature: firstly, that vector product is preserved by rotation in the sense that

$$R(\mathbf{a}) \times R(\mathbf{b}) = R(\mathbf{a} \times \mathbf{b})$$

and secondly, that curl is preserved, in the sense that

$$\nabla \times (R(\mathbf{F})) = R(\nabla \times \mathbf{F}).$$

(Exercise: using linear algebra, representing the rotation R as a 3×3 matrix A which satisfies $A = A^T$ and $\det A = 1$, prove these two identities!)

Back to our problem! Choose R so that $R(1,1,1) = (0,0,\sqrt{3})$. It rotates the annoying cap Σ so that $R(\Sigma)$ is the part of the unit sphere where $z \geq 1/\sqrt{3}$ (this was the distance of the plane P from the origin). When we rotate $\nabla \times \mathbf{F}$ we get $\nabla \times R(\mathbf{F})$, and when we rotate $\mathbf{F} = (x,y,z) \times (1,1,1)$ we get $(x,y,z) \times (0,0,\sqrt{3})$. We have therefore transformed the problem to one where the plane $x + y + z \geq 1$ is replaced by $z \geq 1/\sqrt{3}$ and the vector field $(x,y,z) \times (1,1,1)$ is replaced by $(x,y,z) \times (0,0,\sqrt{3})$.

We still need to solve this new problem, of course! If we end up using one of the 'trick' methods (3 or 4) then this whole 'rotation of the problem' is largely a waste of time, because the method doesn't really depend on the coordinate system; but if we were doing a direct calculation (method 1 and 2) then it will certainly be a lot easier now that we can parametrise using the standard spherical coordinate system.

This rotation trick would really come into its own if the problem were something like "compute the flux through Σ of the electric field generated by a particle of charge 1 placed at $(3,3,3)$". Here there is no Stokes-type shortcut allowing us to avoid actually doing an integral over Σ, but it would be much easier to work out the answer using the rotated problem where Σ has been changed into the surface $z \geq 1/\sqrt{3}$ and the charge is at $(0,0,3\sqrt{3})$.