2012 MT 2 solutions

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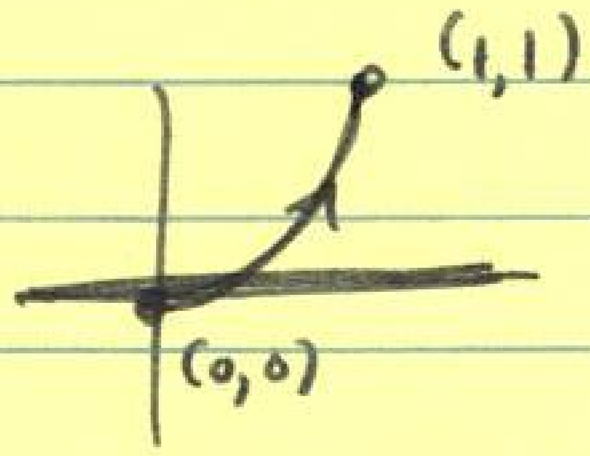


$y^2 = x^3$ $\qquad \displaystyle\int_\gamma x^2 y\, dx - xy\, dy$

The 'obvious' parametrization of the curve is "via the x-coordinate", ie ~~$t \mapsto (t, t^{2/3})$~~

$0 \le t \le 1 \qquad t \mapsto (t, t^{3/2}) \qquad (\text{since } y = x^{3/2})$

This gives you

$$\int_0^1 dt \; \underbrace{\begin{pmatrix} t^2 \cdot t^{3/2} \\ -t \cdot t^{3/2} \end{pmatrix}}_{\begin{pmatrix} x^2 y \\ -xy \end{pmatrix}} \cdot \underbrace{\begin{pmatrix} 1 \\ 3/2\, t^{1/2} \end{pmatrix}}_{\frac{ds}{dt}} = \cdots$$

But I hate fractional powers!

So an easier calculation will result if we use instead $\quad t \mapsto (t^2, t^3) \quad 0 \le t \le 1$

(ie. $x = t^2$, $y = t^3$ does satisfy $y^2 = x^3$).

Because then we get

$$\int_0^1 dt \begin{pmatrix} t^7 \\ -t^5 \end{pmatrix} \cdot \begin{pmatrix} 2t \\ 3t^2 \end{pmatrix} = \int_0^1 (2t^8 - 3t^7)\, dt$$

$$= \frac{2}{9} - \frac{3}{8} = \underline{\underline{\frac{-11}{72}}}$$

NB
- can't use Green's theorem here — not a closed curve!
- You're not expected to find the 'trick' parametrization above, but it's a good example of a "non-standard" one being useful.

② Standard Green's theorem arc:

$$\int_{\gamma} (x-y^2)\,dx + (x^3+y^4)\,dy = \iint_{▨} 3x^2 + 2y \quad dx\,dy$$

$$= \iint 3x^2 \; dx\,dy$$ (second term vanishes by symmetry!)

$= [illegible]\,[x^3]_{-1}^{1} \times$ (y-length)

$= 2 \times 2 = \underline{\underline{4}}$

③ I did this in the review lecture:

using $\underline{\hat{n}} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ we have $\underline{F}\cdot\underline{\hat{n}} = -x^2+y^2+z^2$

so need $\int_{\Sigma} (-x^2+y^2+z^2)\,dA$.

By symmetry (of the sphere) $\int x^2\,dA = \int y^2\,dA = \int z^2\,dA$ ⊛

but $\int (x^2+y^2+z^2)\,dA = \int r^2\,dA = \int 1\,dA = 4\pi$ (on unit sphere)

$\therefore$ each of ⊛ integrals $= \frac{4\pi}{3}$

so $\int (-x^2+y^2+z^2)\,dA = -\frac{4\pi}{3} + \frac{4\pi}{3} + \frac{4\pi}{3} = \underline{\underline{\frac{4\pi}{3}}}$.

($\underline{\underline{\text{Alt}}}$: do $\int_{\Sigma} z^2\,dA = \int_0^{\pi}\int_0^{2\pi} (\cos^2\phi)\cdot\sin\phi\,d\theta\,d\phi$

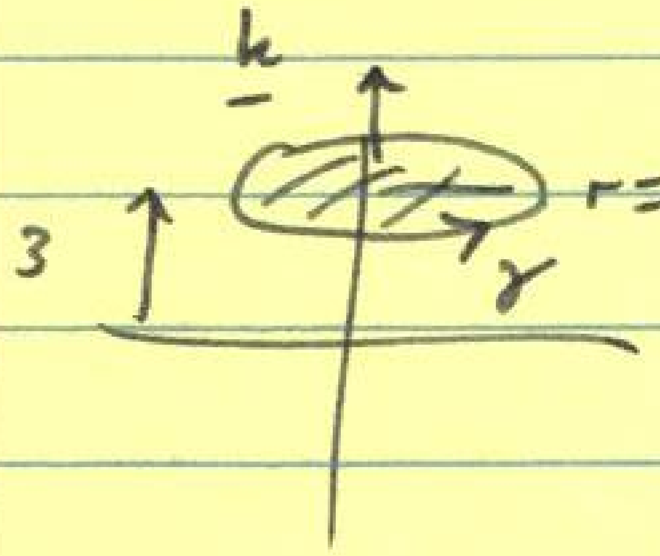
...

)

(You could also compute $\int z^2\,dA$ over a <u>hemisphere</u>, then (by using 'graph' approach, $z = \sqrt{1-x^2-y^2}$) double it!)

④ (There's a typo in the question :

$d\underset{\sim}{S}$ should be $d\underset{\sim}{s}$



By Stokes, $\displaystyle\int_\gamma \underline{F}\cdot d\underline{s} = \int_D (\underline{\nabla}\times\underline{F})\cdot d\underline{S}$

$$= \int (\underline{\nabla}\times\underline{F})\cdot\underline{k}\ dS$$

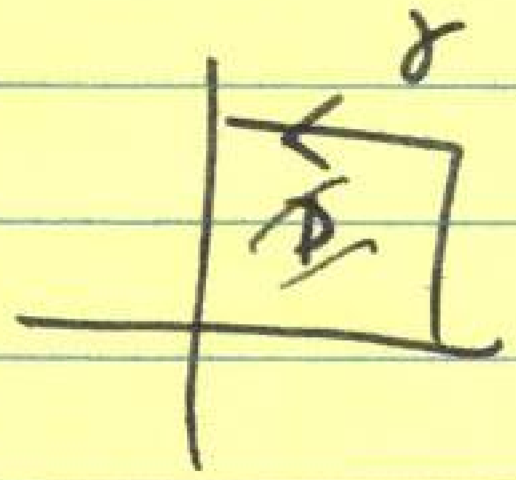
↑ scalar surface integral now

constant unit normal

$\underline{\nabla}\times\underline{F}$ is easy : $\begin{pmatrix}\partial/\partial x\\ \partial/\partial y\\ \partial/\partial z\end{pmatrix}\times\begin{pmatrix}x^2\cos x\\ y^2\cos y\\ xyz\end{pmatrix} = \begin{pmatrix}xz\\ -yz\\ 0\end{pmatrix}$

So we get $\displaystyle\int_{\text{disc}} \begin{pmatrix}xz\\ -yz\\ 0\end{pmatrix}\cdot\begin{pmatrix}0\\ 0\\ 1\end{pmatrix} dS = \underline{\underline{0}}$.

2013 MT 2 solutions

① 

$$\int_\gamma \overbrace{(x^2-y^2)}^{P}\,dx + \overbrace{(x^2+y^2)}^{Q}\,dy$$

$$\underset{\text{(Green's thm)}}{=} \iint_D 2x - (-2y)\;dx\,dy$$

Easiest to think of just $\int_0^1\int_0^1 2x \; dx\,dy$

$= [x^2]_0^1 \times$ length of y-range $= 1$

so $\iint_D (2x+2y)\,dx\,dy = \underline{\underline{2}}$

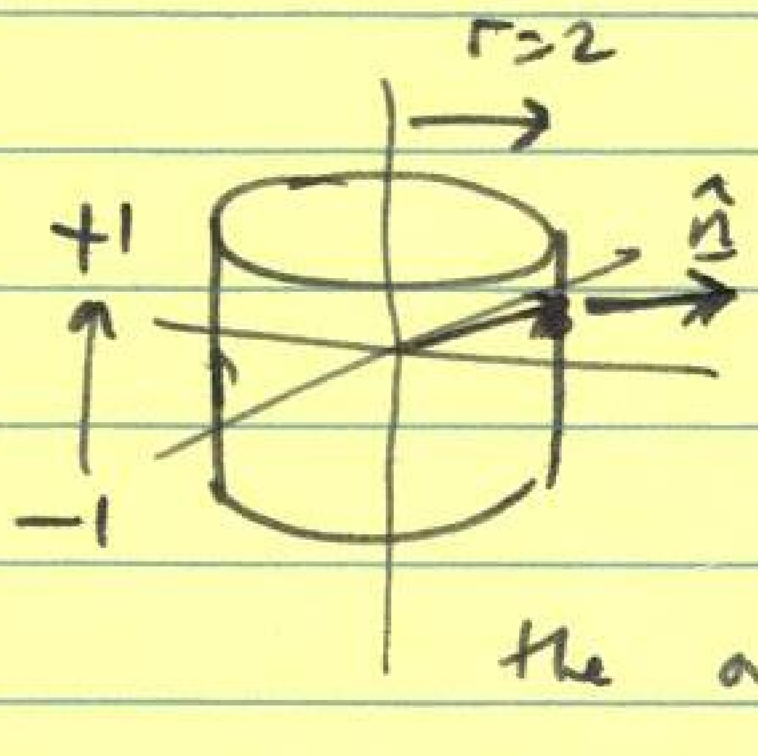
② Change to spherical coordinates, $Z = \cos\phi$

$dA = \sin\phi \, d\theta \, d\phi$

so $\int_\Sigma z^4\,dA = \int_0^{\pi/2} d\phi \int_0^{2\pi} d\theta \;\sin\phi . \cos^4\phi$

$$= 2\pi \times \left[\frac{-1}{5}\cos^5\phi\right]_0^{\pi/2}$$

$$= \underline{\underline{2\pi/5}}$$

③ 

I like to do these using the knowledge of the outward unit normal; at the point (x,y,z), the outward unit normal is $\underline{\hat{n}} = \begin{pmatrix} x/2 \\ y/2 \\ 0 \end{pmatrix}$

(it's really the unit normal at (x,y) on a circle of radius 2,

& with no z-coordinate since it points horizontally).

Thus we have $\int_\Sigma \underline{F}\cdot d\underline{A} = \int_\Sigma (\underline{F}\cdot\hat{\underline{n}})\, dA$

$$= \int_\Sigma \begin{pmatrix}x\\y\\z\end{pmatrix}\cdot\begin{pmatrix}x/2\\y/2\\0\end{pmatrix} dA$$

$$= \int_\Sigma \frac{1}{2}(x^2+y^2)\, dA$$

since $x^2+y^2=4$ is constant $= \int 2\, dA = 2 \times \text{area}$

$$= 2 \times \underset{\substack{\uparrow\\ \text{circumference}}}{4\pi} \times \underset{\substack{\uparrow\\ \text{height}}}{2}$$

(area of cylinder: imagine slicing & unrolling it into a $2\pi r \times h$ rectangle!)

$$\underline{\underline{= 16\pi}}$$

(Alternative: parametrise cylinder via

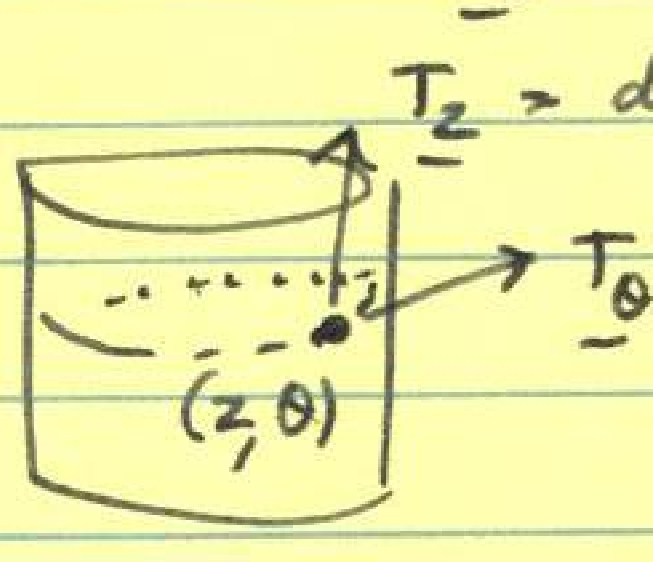
$$(z,\theta) \mapsto (2\cos\theta,\ 2\sin\theta,\ z)$$

$-1 \le z \le +1$

$0 \le \theta \le 2\pi$ & compute $\iint \underline{F}\cdot(\underline{T}_\theta \times \underline{T}_{\phi z})\, d\theta\, d\phi$.

(we want $\underline{T}_\theta \times \underline{T}_z$ to give the outward normal :

$\underline{T}_z$ = direction of z-increasing



= direction of θ increasing

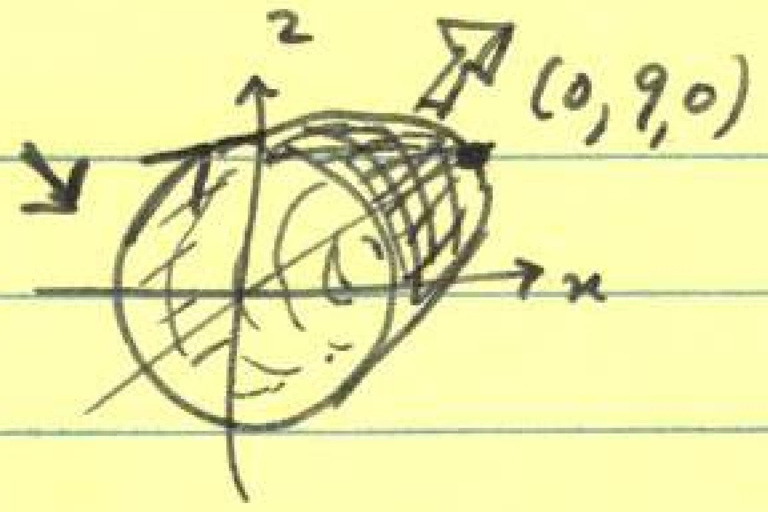
right-hand rule $\Rightarrow$ $\underline{T}_\theta \times \underline{T}_z$ is outwards.

In any case $\underline{F} = \begin{pmatrix}x\\y\\z\end{pmatrix}$ clearly flows out of the cylinder, so a positive value of flux must occur.

④

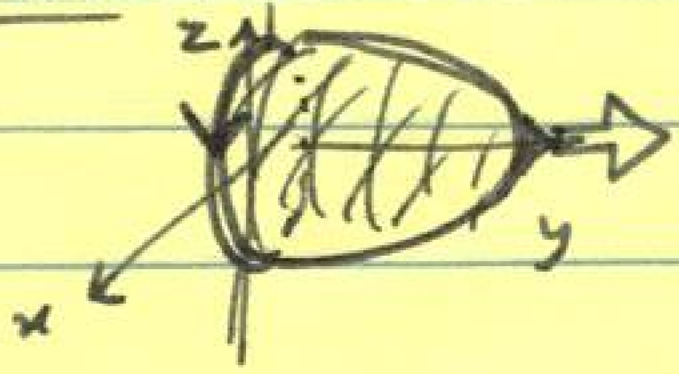
③

circle in (x,z) plane of radius 3



• boundary circle is in fact oriented <u>clockwise</u> as we see it, so as to circulate anti<u>clockwise</u> around the exterior normal vector

(different perspective) ↗



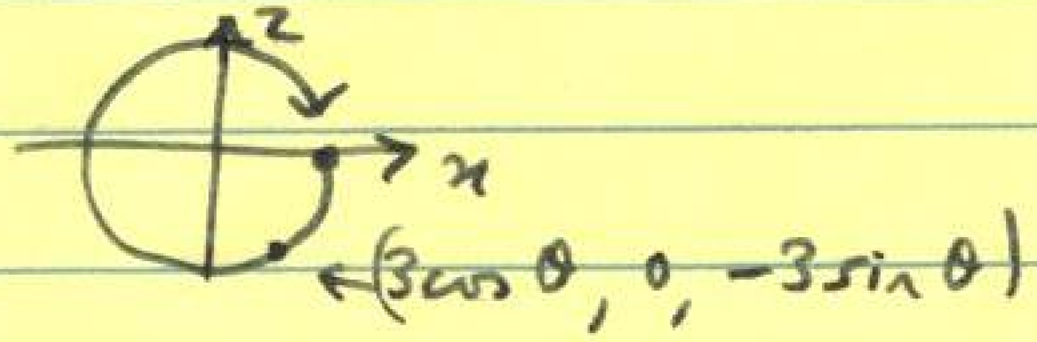
<u>Boring method</u>: use Stokes so

$$\int_{\Sigma} (\underline{\nabla}\times\underline{F})\cdot d\underline{A} = \int_{\partial\Sigma} \underline{F}\cdot d\underline{s}$$

parametrize the boundary circle by

$$\theta \mapsto (3\cos\theta,\ 0,\ -3\sin\theta)$$

[I put a minus in the z-coord to make the clockwise circulation:



– but you could just go anticlockwise & put the minus sign in at the end, explaining why!]

Then compute $\int_0^{2\pi} d\theta\ \underline{F}\cdot\frac{d\underline{s}}{dt}\ dt$

$$= \int_0^{2\pi} \begin{pmatrix} 5(-3\sin\theta) \\ 1 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} -3\sin\theta \\ 0 \\ -3\cos\theta \end{pmatrix} d\theta$$

(when $y=0$, $\underline{F}(x,y,z)$ is $\begin{pmatrix} 5z \\ 1 \\ 0 \end{pmatrix}$, and $z = -3\sin\theta$ on the circle)

$$= \int_0^{2\pi} 45 \sin^2\theta \, d\theta$$

$$= \underline{\underline{45\pi}}. \qquad \Big(\sin^2\theta = \underbrace{\tfrac{1}{2}}_{\text{contributes } \pi \text{ to } \int_0^{2\pi}} \underbrace{(1-\cos 2\theta)}_{\text{contributes 0 (integral over whole cycle average out to 0!)}} \Big)$$

Fun method :

Since we're integrating a curl, we can (by using Stokes' thm twice, essentially) change the surface to another surface with the same oriented boundary, such as the flat disc in the (x,z) plane, of radius 3, whose outward unit normal is just $\underset{\sim}{j}$.

Thus we want to compute $\int_{\text{disc}} (\nabla \times \underline{F}) \cdot \underline{j} \; dA$

The $\underline{j}$-component of the curl is $- \begin{vmatrix} \partial/\partial x & 2xyz + 5z \\ \partial/\partial z & x^2 y \end{vmatrix}$

$$= - \big(2xy - (2xy + 5) \big) = +5$$

Thus we have simply $\int_{\substack{\text{disc of} \\ \text{radius } 3}} 5 \, dA = 5 \times \text{area}$

$$= \underline{\underline{45\pi}}$$

(Mike forgot the radius 3 in his solution)

20E Fall 14, MT 2 solutions

(!) Sorry the exam was hard — or perhaps just too long. (I had originally put one more question on it, with hints for #3, but decided to shorten it & remove the hint, which I feared would confuse some people (there are at least two ways to do this question).

Anyway, on the final you'll have three hours for 8 questions (some of which will be relatively easy) & there's plenty of time to practice between now & then. We've more or less covered all the 'calculational' stuff in the course at this point; what remains is additional *conceptual* understanding of how it all fits together, & relates to physics.

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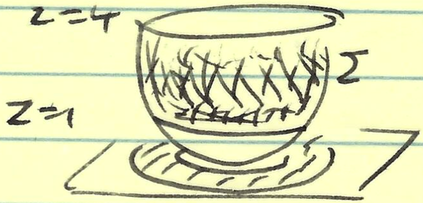


$z = x^2 + y^2$

The part we want lives above the annulus between $r=1$ & $r=2$ in the plane. So we can view it as a *graph* and use the standard formula for $\underline{T_u} \times \underline{T_v}$ ...

So let $D = \{ (u,v) : 1 \le u^2 + v^2 \le 4 \}$
and parametrize Σ via $(u,v) \longmapsto (u, v, u^2+v^2)$

Then $\underline{T_u} \times \underline{T_v} = \begin{pmatrix} -2u \\ -2v \\ 1 \end{pmatrix}$

and so $\| \underline{T_u} \times \underline{T_v} \| = \sqrt{1+4u^2+4v^2}$

We need to compute $\iint_D du\, dv \,.\, \sqrt{1+4u^2+4v^2}$

so change to polar coordinates, $du\,dv = r\,dr\,d\theta$

giving $\int_0^{2\pi} d\theta \int_1^2 dr \,.\, r \,.\, \sqrt{1+4r^2}$

$$= 2\pi \times \left[(1+4r^2)^{3/2} \times \underbrace{\frac{2}{3} \times \frac{1}{8}} \right]_1^2$$

(I'm just putting in 'correction factors' here – I tend to do this kind of integral by "guess & modify" rather than formal integration by substitution !)

$$= \underline{\underline{\frac{\pi}{6} \left(17^{3/2} - 5^{3/2} \right)}}$$

② . The fact that the vector field is so ridiculous strongly suggests we should apply Stokes' theorem! So let's straight off compute $\underline{\nabla} \times \underline{F}$, hoping it will be simpler than $\underline{F}$, and easier to integrate!

$$\underline{\nabla} \times \underline{F} = \begin{pmatrix} \partial_x \\ \partial_y \\ \partial_z \end{pmatrix} \times \begin{pmatrix} 3\cos x + z \\ 5x - e^y \\ z^4 - 3y \end{pmatrix} = \begin{pmatrix} -3 \\ -(-1) \\ 5 \end{pmatrix} = \begin{pmatrix} -3 \\ 1 \\ 5 \end{pmatrix} \quad ③$$

This is <u>much</u> nicer than the horrible $\underline{F}$!

So let's use $\displaystyle\int_C \underline{F} \cdot d\underline{s} = \int_D (\underline{\nabla} \times \underline{F}) \cdot d\underline{S}$

where D is the disc in the plane $2x + 2y + z = 5$ whose boundary is C.

[we could take <u>any other</u> surface Σ whose boundary is C, but the disc D is the simplest because its normal vector will be <u>constant</u>.]

Since the unit normal to the plane $2x + 2y + z = 5$ is $\underline{\hat{n}} = \frac{1}{3}\begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix}$ [$\begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix}$ is <u>a</u> normal, but its magnitude is $\sqrt{1+2^2+2^2} = 3$, so I rescale to get unit vector]

we can use

$$\int_D (\underline{\nabla} \times \underline{F}) \cdot d\underline{S} = \int_D \underbrace{(\underline{\nabla} \times \underline{F}) \cdot \underline{\hat{n}}}_{\text{Scalar function}} \, dS \;\nwarrow \text{scalar integral}$$

$$= \int_D \begin{pmatrix} -3 \\ 1 \\ 5 \end{pmatrix} \cdot \frac{1}{3}\begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix} dS$$

$$= \int_D \frac{1}{3} \, dS = \frac{1}{3} \text{ area } (D)$$

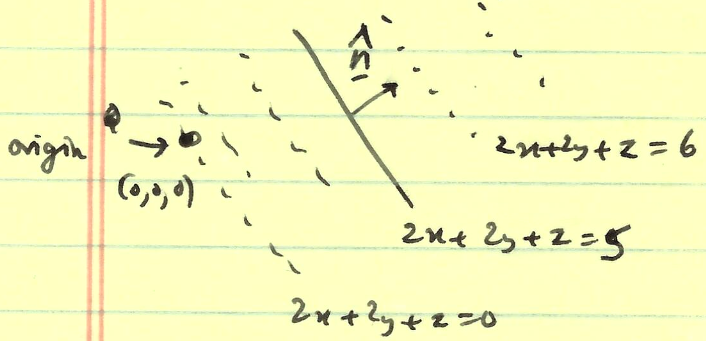
$$= \pi/3 \quad \text{since } D \text{ has radius } 1$$

(we don't care where the centre is!)

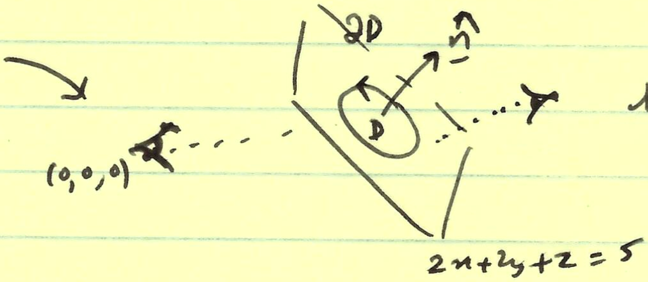
- Now the only remaining problem is the <u>orientation</u>.

I have computed the flux integral $\displaystyle\int_D (\underline{\nabla} \times \underline{F}) \cdot d\underline{S}$

as though D were oriented using the normal $\hat{\underline{n}} = \frac{1}{3}\begin{pmatrix}2\\2\\1\end{pmatrix}$. This is in fact pointing <u>away</u> from the origin, since moving in the direction of $\hat{\underline{n}}$ will take you to points where $2x+2y+z > 5$, yet the origin lies on $2x+2y+z=0$.



So with this orientation of D, the boundary circle would be oriented <u>anticlockwise</u> if you look from the <u>non</u>-origin side; ~~&~~ which is indeed <u>clockwise</u> if you look from the other (origin) side.



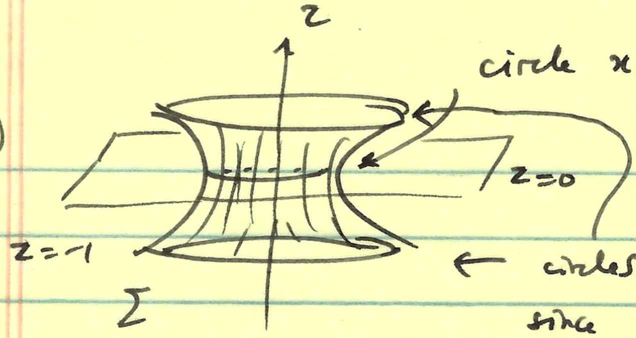
these crappy drawings were meant to be "eyes" looking towards D!

The conclusion is that this was the <u>correct</u> choice of orientation of D to make ∂D oriented clockwise as seen from the origin.

Hence we got the correct answer:

$$\underline{\underline{\int_C \underline{F}\cdot d\underline{s} = +\frac{\pi}{3}}}$$

③



circle $x^2+y^2=1$ in (xy) plane.

← circles of radius $\sqrt{2}$, since $x^2+y^2=1+(\pm 1)^2$ on these boundaries of the surface.

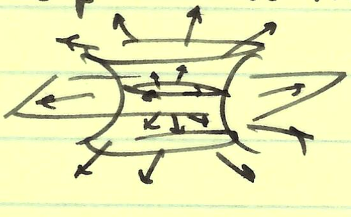
There are ~~two~~ three possible methods here, (at least):

(a) Parametrise the whole surface Σ: since it's rotationally symmetric about the z-axis, it is a good idea to introduce an angular 'θ' coordinate running from 0 to 2π in the normal way. If we know z and θ then we know exactly where we are: so this gives us a parametrisation $(\theta, z) \longmapsto (r\cos\theta, r\sin\theta, z)$ but where the radius r is determined by z according to $x^2+y^2=1+z^2$, ie. $r^2=1+z^2$, ie $r=+\sqrt{1+z^2}$ (since $r \geq 0$ in this type of polar coordinate system!)

Thus $(\theta, z) \longmapsto (\sqrt{1+z^2}\cos\theta, \sqrt{1+z^2}\sin\theta, z)$ will work, for $0 \leq \theta \leq 2\pi$, $-1 \leq z \leq +1$.

(b) If you introduce θ but then try to write everything in terms of r & θ (instead of z & θ) you have the problem that $z = \pm\sqrt{r^2-1}$ is not completely determined when you know r & θ. If you choose $z = \cancel{\pm\sqrt{r^2-1}}$ you are parametrising
$$z = +\sqrt{r^2-1}$$
only the upper half of the hyperboloid $(z \geq 0)$.

However, this is not too serious a problem, since the whole situation is symmetrical under reflection in the x-y plane (i.e. the hyperboloid is symmetric, & the vector field $\underline{F}$ also reflects) Therefore the flux over each half will be the same, and we can just integrate over one half & double the answer.



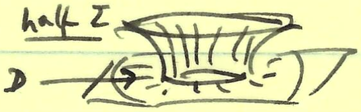
$\updownarrow$ reflect takes $\Sigma \to \Sigma$

and F to itself too

So this second method is to parametrize via $(\theta, r) \longmapsto (r\cos\theta, r\sin\theta, z=\sqrt{r^2-1})$ (the top half) for $0 \le \theta \le 2\pi$, $1 \le r \le \sqrt{2}$

& double the answer at the end.

(c) If we're happy to integrate over half of it and double the result, we can view it as a graph!



It lives over

$D = 1 \le u^2+v^2 \le 2$

so we could use

$$(u,v) \longmapsto (u, v, \sqrt{u^2+v^2-1})$$

for $(u,v) \in D$

(It's fairly clear that once we compute $\underline{F} \cdot (T_{\underline{u}} \times T_{\underline{v}})$ that we'll have to convert to polar coordinates (r, θ) to do the integral over the annulus D)

Computations !

(a) $(\theta, z) \longmapsto (\sqrt{1+z^2}\cos\theta,\ \sqrt{1+z^2}\sin\theta,\ z)$ ⊛

$$\underline{T_\theta} = \begin{pmatrix} -\sqrt{1+z^2}\cdot\sin\theta \\ \sqrt{1+z^2}\ \cos\theta \\ 0 \end{pmatrix} \qquad \underline{T_z} = \begin{pmatrix} \frac{z}{\sqrt{1+z^2}}\ \cos\theta \\ \frac{z}{\sqrt{1+z^2}}\ \sin\theta \\ 1 \end{pmatrix}$$

$$\left(\text{note : } \frac{d}{dz}\left(\sqrt{1+z^2}\right) = \frac{z}{\sqrt{1+z^2}} \right) \nearrow$$

$$\therefore \quad \underline{T_\theta} \times \underline{T_z} = \begin{pmatrix} \sqrt{1+z^2}\ \cos\theta \\ \sqrt{1+z^2}\ \sin\theta \\ -z \end{pmatrix} \leftarrow (\text{used } \cos^2+\sin^2=1)$$

$$\Rightarrow \underline{F}\cdot(\underline{T_\theta}\times\underline{T_z}) = \begin{pmatrix} \sqrt{1+z^2}\cos\theta \\ \sqrt{1+z^2}\sin\theta \\ z \end{pmatrix} \cdot \begin{pmatrix} \sqrt{1+z^2}\cos\theta \\ \sqrt{1+z^2}\sin\theta \\ -z \end{pmatrix} = (1+z^2)-z^2 \ \underline{\underline{=1}} \ !$$

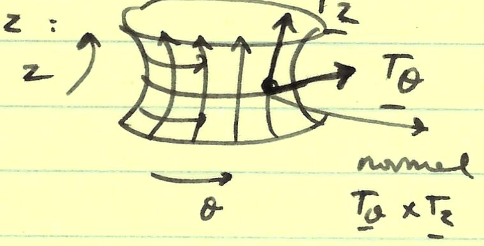
at (x,y,z), $\underline{F}$ is $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ so I just used original param. formula ⊛ to turn it into θ & z coords

(used $\cos^2+\sin^2=1$ once more!)

Therefore our integral is merely $\displaystyle\int_0^{2\pi} d\theta \int_{-1}^{+1} dz \cdot 1 = \underline{\underline{4\pi}}$!

Check orientation! (i) You can see that at the point (x,y,z) on the hyperboloid, $\underline{T_\theta}\times\underline{T_z}$ is actually the vector $\begin{pmatrix} x \\ y \\ -z \end{pmatrix}$! This clearly points outwards as required.

(ii) Alternative is to look at the picture, & think about lines of increasing θ & increasing z : it's clear that $\underline{T_\theta}\times\underline{T_z}$ points outwards from this.



(b) This way we have <u>half</u> the surface, parametrized

$$(\theta, r) \longmapsto \left(r\cos\theta,\ r\sin\theta,\ \sqrt{r^2-1}\right) \qquad \left(\begin{matrix} 0 \le \theta \le 2\pi \\ 1 \le r \le \sqrt{2} \end{matrix}\right) \quad ⊛$$

so $$\underline{T_\theta} = \begin{pmatrix} -r\sin\theta \\ r\cos\theta \\ 0 \end{pmatrix} \qquad \underline{T_r} = \begin{pmatrix} \cos\theta \\ \sin\theta \\ \frac{r}{\sqrt{r^2-1}} \end{pmatrix}$$

$$\underline{T_\theta} \times \underline{T_r} = \begin{pmatrix} r^2\cos\theta/\sqrt{r^2-1} \\ r^2\sin\theta/\sqrt{r^2-1} \\ -r \end{pmatrix}$$

$$\Rightarrow \underline{F}\cdot(\underline{T_\theta}\times\underline{T_r}) = \begin{pmatrix} r\cos\theta \\ r\sin\theta \\ \sqrt{r^2-1} \end{pmatrix} \cdot \begin{pmatrix} r^2\cos\theta/\sqrt{r^2-1} \\ r^2\sin\theta/\sqrt{r^2-1} \\ -r \end{pmatrix}$$

once again, $\underline{F}$ is just $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ written in terms of θ & r via ⊛

$$= \frac{r^3}{\sqrt{r^2-1}} - r\sqrt{r^2-1} \qquad (\text{used } c^2+s^2=1, \text{ again!})$$

$$= \frac{r^3 - r(r^2-1)}{\sqrt{r^2-1}}$$

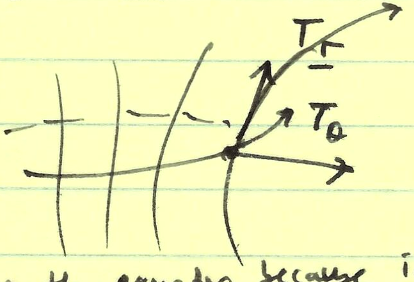
$$= \frac{r}{\sqrt{r^2-1}}$$

so we get $$\int_0^{2\pi} d\theta \int_1^{\sqrt{2}} dr\, \frac{r}{\sqrt{r^2-1}} = 2\pi \times \left[\sqrt{r^2-1}\right]_1^{\sqrt{2}}$$

$$= 2\pi \times (1-0) = 2\pi$$

but this is only <u>half</u> the answer $\therefore$ really $\underline{\underline{4\pi}}$

(orientation check:



$\underline{T_r}$ still points <u>up</u> on the equator because increasing r from 1 to $\sqrt{2}$ moves "radially" up $\underline{z}$)

(c) Plug in the standard $\underline{T_u} \times \underline{T_v} = \begin{pmatrix} -f_u \\ -f_v \\ 1 \end{pmatrix}$ formula!

$$(u, v) \longmapsto \left(u, v, \sqrt{u^2+v^2-1}\right)$$

so $\underline{T_u} \times \underline{T_v} = \begin{pmatrix} -u/\sqrt{u^2+v^2-1} \\ -v/\sqrt{u^2+v^2-1} \\ 1 \end{pmatrix}$

$\underline{F} \cdot (\underline{T_u} \times \underline{T_v})$ is $\begin{pmatrix} u \\ v \\ \sqrt{u^2+v^2-1} \end{pmatrix} \cdot \begin{pmatrix} -u/\sqrt{u^2+v^2-1} \\ -v/\sqrt{u^2+v^2-1} \\ 1 \end{pmatrix}$

$$= \frac{-(u^2+v^2)}{\sqrt{u^2+v^2-1}} + \sqrt{u^2+v^2-1}$$

When we change to polar coordinates we now get an integral identical to that of (b) $_*$ (almost!):

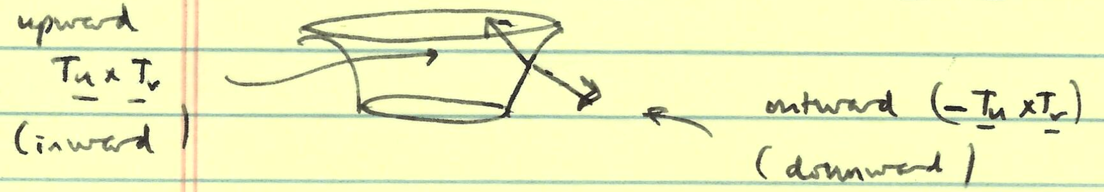
$$\iint_D du\,dv \left(\frac{-u^2+v^2}{\sqrt{u^2+v^2-1}} + \sqrt{u^2+v^2-1}\right)$$

$$= \iint_{\cancel{\text{mod}}}^{\sqrt{2}\ \cancel{2\pi}} r\,dr\,d\theta \cdot \left(\frac{-r^2}{\sqrt{r^2-1}} + \sqrt{r^2-1}\right)$$

$$= \begin{pmatrix}\text{common} \\ \text{denominator} \\ \text{as before}\end{pmatrix} = \int_1^{\sqrt{2}} \int_0^{2\pi} d\theta \int dr \left(\frac{-r}{\sqrt{r^2-1}}\right) = \underline{\underline{-2\pi}}$$

We double the answer, to $\underline{\underline{-4\pi}}$

But the sign is <u>wrong</u> this time, because the "graph" parametrization corresponds to <u>upward</u> normal, which is <u>inward</u> not <u>outward</u>:



$\underline{\underline{\text{so}}}$ $\underline{\underline{+4\pi}}$!

2016 MT2 Solutions

①

Method 1 : parametrise the hemisphere as a graph ($z = \sqrt{1-x^2-y^2}$) using unit disc D in (u,v)-plane, and quoting the formula for $\underline{T}_u \times \underline{T}_v = \begin{pmatrix} -\partial g/\partial u \\ -\partial g/\partial v \\ 1 \end{pmatrix}$.

Since $g(u,v) = \sqrt{1-u^2-v^2}$ this gives

$$\underline{T}_u \times \underline{T}_v = \begin{pmatrix} u/\sqrt{1-u^2-v^2} \\ v/\sqrt{1-u^2-v^2} \\ 1 \end{pmatrix}$$

Thus we get

$$\int_\Sigma \underline{F} \cdot d\underline{A} = \iint_D \begin{pmatrix} v \\ u \\ \sqrt{1-u^2-v^2} \end{pmatrix} \cdot \begin{pmatrix} u/\sqrt{1-u^2-v^2} \\ v/\sqrt{1-u^2-v^2} \\ 1 \end{pmatrix} du\,dv$$

$$= \iint_D \left(\frac{2uv}{\sqrt{1-u^2-v^2}} + \sqrt{1-u^2-v^2} \right) du\,dv$$

The first piece vanishes by symmetry (in either the u-direction or v-direction)!

The second bit becomes nice when we change to polar coords

$$= \int_0^{2\pi}\int_0^1 \sqrt{1-r^2}\; r\,dr\,d\theta = \left[\frac{-1}{2} \cdot \frac{2}{3} (1-r^2)^{3/2} \right]_0^1 \times 2\pi$$

$$\underline{\underline{\frac{2\pi}{3}}}$$

Method 2 Use spherical coordinate parametrisation.

This is similar to the above, but even messier!

with parametrization
$$x = \cos\theta \sin\phi \qquad 0 \leq \theta \leq 2\pi$$
$$y = \sin\theta \sin\phi \qquad 0 \leq \phi \leq \pi/2$$
$$z = \cos\phi$$

we have $\underline{T}_\theta \times \underline{T}_\phi = -\sin\phi \begin{pmatrix} \cos\theta \sin\phi \\ \sin\theta \sin\phi \\ \cos\phi \end{pmatrix}$

(~~so~~ This is the wrong orientation (stupid book coordinate system) so we should actually change the sign to make sure that we are getting an <u>outward</u> normal vector ... technically we should use $\underline{T}_\phi \times \underline{T}_\theta$ not $\underline{T}_\theta \times \underline{T}_\phi$.)

Thus
$$\int_\Sigma \underline{F} \cdot d\underline{A} = \int_0^{2\pi} d\theta \int_0^{\pi/2} d\phi \underbrace{\begin{pmatrix} \sin\theta \sin\phi \\ \cos\theta \sin\phi \\ \cos\phi \end{pmatrix}}_{\underline{F}} \cdot \underbrace{\begin{pmatrix} \cos\theta \sin\phi \\ \sin\theta \sin\phi \\ \cos\phi \end{pmatrix} \sin\phi}_{\underline{T}_\phi \times \underline{T}_\theta}$$

$$= \int_0^{2\pi} d\theta \int_0^{\pi/2} d\phi \left(\underset{\uparrow}{\sin 2\theta} \sin^3\phi + \cos^2\phi \sin\phi \right)$$

$\sin 2\theta = 2 \sin\theta \cos\theta$

The integral $\int_0^{2\pi} d\theta \sin 2\theta$ is 0 so the first term dies, & we're left with just
$$2\pi \times \left[\frac{-1}{3} \cos^3\phi \right]_0^{\pi/2} = \underline{\underline{\frac{2\pi}{3}}}$$

<u>Method 3</u> Use the fact that the area element on the sphere can be written down in terms of the ~~radial~~ unit normal vector quite easily:
$$\underset{\sim}{dA} = \underline{\hat{n}} \, dA = \begin{pmatrix} x \\ y \\ z \end{pmatrix} dA$$

so that $\displaystyle\int_\Sigma \underline{F}\cdot d\underline{A} = \int_\Sigma \begin{pmatrix} y \\ x \\ z \end{pmatrix}\cdot\begin{pmatrix} x \\ y \\ z \end{pmatrix} dA$

$$= \int_\Sigma (2xy + z^2)\, dA$$

The first integral is 0 by symmetry in x (or y) direction.

The second can be done by the trick that

$$\int_{\text{hemisphere}} z^2\, dA = \frac{1}{2}\int_{\text{sphere}} z^2\, dA = \frac{1}{6}\int_{\text{sphere}} (x^2+y^2+z^2)\, dA$$

$$= \frac{1}{6}(\text{area of sphere})$$

$$= \underline{\underline{2\pi/3}}$$

② The surface is nasty, but its boundary is easy (the circle of radius π in the plane $z = \cos\pi^2$) so we probably should apply Stokes' thm:

Method 1

$$\int_\Sigma (\underline{\nabla}\times\underline{F})\cdot d\underline{A} = \int_{\partial\Sigma} \underline{F}\cdot d\underline{s}$$

Parametrize the curve using

$x = \pi\cos\theta$ $\quad 0 \le \theta \le 2\pi$

$y = \pi\sin\theta$

$z = \cos\pi^2$

then $\displaystyle\frac{d\underline{s}}{d\theta} = \begin{pmatrix} -\pi\sin\theta \\ \pi\cos\theta \\ 0 \end{pmatrix}$

$\underline{F}\downarrow$

so we get $\displaystyle\int_0^{2\pi} d\theta \begin{pmatrix} -\pi\sin\theta \\ \pi\cos\theta \\ \cos\pi^2 \end{pmatrix}\cdot\begin{pmatrix} -\pi\sin\theta \\ \pi\cos\theta \\ 0 \end{pmatrix}$.

$$= \int_0^{2\pi} \pi^2(\sin^2\theta + \cos^2\theta)\, d\theta$$

$$= \underline{\underline{2\pi^3}}$$

Method 2 is to try directly: what make this easy is the fact that

$$\underline{\nabla} \times \underline{F} = \begin{pmatrix} \partial/\partial x \\ \partial/\partial y \\ \partial/\partial z \end{pmatrix} \times \begin{pmatrix} -y \\ x \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix}$$

so if we parametrise as a graph

$$(u,v) \longmapsto (u, v, \cos(u^2+v^2))$$

domain = disc of radius ~~unit disc~~ π in (u,v) plane D

$\swarrow$ $\underline{T}_u \times \underline{T}_v$ – we don't need the x & y components!

we get $$\iint_D \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} \sim \\ \sim \\ 1 \end{pmatrix} du\,dv$$

$$= \int_D 2 \, dA = 2 . \text{area}(D) = \underline{\underline{2\pi^3}}$$

③ Try computing $\int x \, dy$ around the curve:

this becomes (via parametrisation, using $\frac{dy}{dt} = 3\sin^2 t \cos t$)

$$\int_0^{2\pi} \cos^3 t \, . \, 3 \sin^2 t \cos t \; dt$$

$$= 3 \int_0^{2\pi} \cos^4 t \sin^2 t \; dt$$

(using trig identities) $$= \frac{3}{8} \int_0^{2\pi} (1+\cos 2t)^2 (1-\cos 2t) \; dt$$

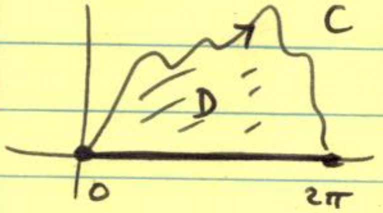
$$= \frac{3}{8} \int_0^{2\pi} \left(1 + \cancel{\cos 2t} - \cos^2 2t - \cancel{\cos^3 2t}\right) dt$$

(two terms vanish because they are odd functions over this range)

$$= \frac{3}{8} \int_0^{2\pi} \left(1 - \tfrac{1}{2}(1+\cancel{\cos 4t})\right) dt = \frac{3}{8} \int_0^{2\pi} \tfrac{1}{2} \, dt = \underline{\underline{\frac{3\pi}{8}}}.$$

Solutions for 20E MT #2, 2017

①



We want to integrate either $\int x\,dy$ or $\int -y\,dx$

anticlockwise round the boundary.

I'm going to start by just computing $\int_C -y\,dx$ with the natural "left-to-right" parametrization of C, then adjust afterwards.

Since on C, $\underline{s}(t) = \begin{pmatrix} t-\sin t \\ 1-\cos t \end{pmatrix}$ we get $\frac{d\underline{s}}{dt} = \begin{pmatrix} 1-\cos t \\ \sin t \end{pmatrix}$

$$\therefore \int_C -y\,dx = \int_0^{2\pi} \underbrace{-(1-\cos t)}_{-y}\underbrace{(1-\cos t)}_{\frac{dx}{dt}}\,dt$$

$$= \int_0^{2\pi} -1 + 2\cos t - \cos^2 t\,dt$$

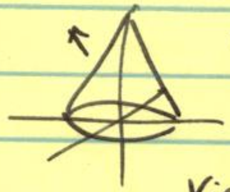
$$= -2\pi + 0 - \int_0^{2\pi} \tfrac{1}{2}(1+\cos 2t)\,dt$$

$$= -2\pi - \left[\frac{t}{2} + \frac{1}{4}\sin 2t\right]_0^{2\pi}$$

$$= -2\pi - \pi = \underline{-3\pi}.$$

Similarly, $\int -y\,dx$ along the x-axis from 0 to 2π is ZERO since $y \equiv 0$ on the axis. Therefore the integral $\int -y\,dx$ anticlockwise around all of ∂D is $-(-3\pi) = \underline{\underline{+3\pi}}$

[Note: If you use $\int x\,dy$ you get a slightly more difficult integral: $\int_C x\,dy = \int_0^{2\pi} (t-\sin t)\cdot\sin t\,dt$ which needs integration by parts. That's why I chose $-y\,dx$.]

②  $z = 1 - \sqrt{x^2+y^2}$. $\underline{F} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ ②

View this as a 'graph' surface $z = g(u,v)$. Then we can easily get

$$\underline{T_u} \times \underline{T_v} = \begin{pmatrix} -g_u \\ -g_v \\ 1 \end{pmatrix} = \begin{pmatrix} u/\sqrt{u^2+v^2} \\ v/\sqrt{u^2+v^2} \\ 1 \end{pmatrix}$$

and so

$$\int_\Sigma \underline{F} \cdot d\underline{A} = \iint_{\substack{\text{unit} \\ \text{disc}}} du\,dv \begin{pmatrix} u \\ v \\ 1-\sqrt{u^2+v^2} \end{pmatrix} \cdot \begin{pmatrix} u/\sqrt{u^2+v^2} \\ v/\sqrt{u^2+v^2} \\ 1 \end{pmatrix}$$

$$= \iint_{\text{unit disc}} du\,dv \left(\frac{u^2+v^2}{\sqrt{u^2+v^2}} + 1 - \sqrt{u^2+v^2} \right)$$

$$= \iint_{\text{unit disc}} du\,dv \,.\, 1 \quad = \text{area of unit disc} \quad = \underline{\underline{\pi}}$$

(We can check this another way. The flux of $\underline{F} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ out of the bottom of the cone – the unit disc in the xy-plane – is clearly zero, since $\underline{F}$ is <u>tangent</u> to that disc.

$$\therefore \int_{\cancel{\text{cone}}\,\Sigma} \underline{F} \cdot d\underline{A} = \text{flux out of } \underline{\text{solid}} \text{ cone} = \int_{\substack{\text{solid} \\ \text{cone}}} (\underline{\nabla} \cdot \underline{F})\, dV$$

but $\underline{\nabla} . \underline{F} = 3$, so this is $\displaystyle\int_{\text{solid cone}} 3\, dV = 3 \times \text{vol cone}$

$$= 3 \times \tfrac{1}{3} \text{ base} \cdot \text{height}$$

$$= 3 \times \tfrac{1}{3} \times \pi \times 1$$

$$= \underline{\underline{\pi}}$$

)

③ Use the 'graph' picture: $z = g(u,v) = -3u^2 + 8uv + 3v^2$

Jacobian factor for area is $\sqrt{1 + g_u^2 + g_v^2}$

$$= \sqrt{1 + (-6u+8v)^2 + (8u+6v)^2}$$

$$= \sqrt{1 + 100u^2 + 100v^2}$$

which we integrate over the unit disc:

$$\iint_{\text{unit disc}} du\,dv\,\sqrt{1+100u^2+100v^2} \underset{\substack{\text{change}\\ \text{to polar}}}{=} \iint r\,dr\,d\theta \,.\, \sqrt{1+100r^2}$$

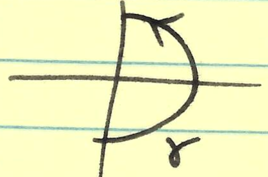
$$= 2\pi \,.\, \int_0^1 r\sqrt{1+100r^2}\,dr$$

$$= 2\pi \,.\, \left[\frac{2}{3}\frac{(1+100r^2)^{3/2}}{200}\right]_0^1$$

$$\underline{\underline{= \frac{\pi}{150}\left(101^{3/2} - 1\right)}}$$

20E Fall 2017 Midterm 2 Solution

①



Parametrise

$$-\frac{\pi}{2} \le \theta \le \frac{\pi}{2} \longrightarrow \underline{s}(\theta) = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}$$

$$\underline{F}(x,y) = \begin{pmatrix} y^2 \\ x \end{pmatrix} \qquad \frac{d\underline{s}}{d\theta} = \begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix}$$

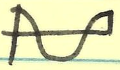
$$\text{so } \int_\gamma \underline{F}\cdot d\underline{s} = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \begin{pmatrix} \sin^2\theta \\ \cos\theta \end{pmatrix} \cdot \begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix} d\theta$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (-\sin^3\theta)\, d\theta + \int_{-\frac{\pi}{2}}^{\pi/2} \cos^2\theta \, d\theta$$

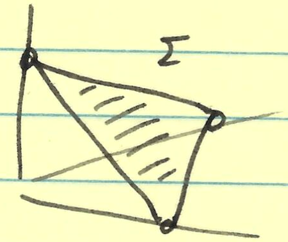
$$= 0 \left(\begin{array}{c} \sin^3\theta \text{ is an} \\ \text{odd function!} \end{array} \right) + \int_{-\frac{\pi}{2}}^{\pi/2} \frac{1}{2}(1 + \underset{\circledast}{\cos 2\theta})\, d\theta$$

(alternative: $\sin^3\theta = \sin\theta \cdot \sin^2\theta$
$= \sin\theta\,(1-\cos^2\theta)$
makes it easy to integrate)

$$= \left[\frac{\theta}{2}\right]_{-\pi/2}^{\pi/2} = \underline{\underline{\frac{\pi}{2}}}.$$

($\circledast$: second part vanishes because we're integrating $\cos 2\theta$ from $\theta = -\frac{\pi}{2}$ to $+\pi/2$; it goes through the whole cycle  so integral is zero!)

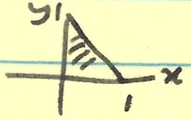
②



Σ is contained in the plane

$$x + y + z = 1$$

so we can view it as the graph of the function $z = 1 - x - y$ over the unit triangle in the xy plane



(which we can integrate over by slicing, x goes $0 \to 1$,
y goes $0 \to 1-x$)

Using the standard graph parametrization ($x = u,\ y = v,\ z = 1-u-v$) we get

$$\int_\Sigma yz\,dA = \int_0^1 du \int_0^{1-u} dv \,.\, v.(1-u-v).\sqrt{3}$$

where the $\sqrt{3}$ comes from $\|\underline{T_u} \times \underline{T_v}\| = \left\|\begin{matrix} -\partial z/\partial u \\ -\partial z/\partial v \\ 1 \end{matrix}\right\| = \left\|\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}\right\|$.

$$= \sqrt{3}. \int_0^1 du \left[\frac{v^2}{2}(1-u) - \frac{v^3}{3}\right]_0^{v=1-u}$$

$$= \sqrt{3} \int_0^1 du \,.\, \frac{(1-u)^3}{6}$$

$$= \frac{\sqrt{3}}{6}\left[\frac{(1-u)^4}{-4}\right]_0^1 = \underline{\underline{\frac{\sqrt{3}}{24}}}$$

③ Using the standard graph parametrization, ($z = uv$)

$$\int_\Sigma \underline{F} \cdot d\underline{A} = \int_0^1 du \int_0^1 dv \underbrace{\begin{pmatrix} 1 \\ u^2 \\ u.v.uv \end{pmatrix}}_{\underline{F}} \cdot \underbrace{\begin{pmatrix} -v \\ -u \\ 1 \end{pmatrix}}_{\underline{T_u} \times \underline{T_v}} = \begin{pmatrix} -\partial z/\partial u \\ -\partial z/\partial v \\ 1 \end{pmatrix}$$

$$= \int_0^1 du \int_0^1 dv \left(-v \mp u^3 + u^2v^2\right)$$

$$= \int_0^1 du \left[-\frac{1}{2} \mp u^3 + \frac{u^2}{3}\right] \qquad \left(\text{I used } \int_0^1 v^n\,dv = \frac{1}{n+1}!\right)$$

$$= -\frac{1}{2} \cancel{+} -\frac{1}{4} + \frac{1}{9} = \cancel{\frac{31}{36}}\ \underline{\underline{\frac{-23}{36}}}$$