Vector Calculus 20E, Spring 2012, Lecture B, Final exam

Three hours, eight problems. No calculators allowed.

Please start each problem on a new page.

You will get full credit only if you show all your work clearly.

Simplify answers if you can, but don't worry if you can't!

1. Let γ be the ellipse $x^2 + 4y^2 = 4$, oriented anticlockwise. Compute

$$\int_\gamma (4y - 3x)dx + (x - 4y)dy$$

2. Find the integral $\int_\gamma \mathbf{F}.d\mathbf{s}$ where $\mathbf{F} = y\mathbf{i} + x\mathbf{j} + z\mathbf{k}$ and the curve γ is the part of the parabola $z = x^2, y = 0$ going from $x = -1$ to $x = 2$.

3. Find the integral $\int_R xyz\, dS$, where R is the rectangle in $\mathbb{R}^3$ whose vertices are the points $(0, 0, 0), (1, 0, 0), (0, 1, 1), (1, 1, 1)$.

4. Find the area of the surface Σ in $\mathbb{R}^3$ described by

$$(u \cos v, u \sin v, u^2) \qquad 0 \le u \le 2 \quad 0 \le v \le 2\pi.$$

5. Find the flux $\int_\Sigma \mathbf{F}.d\mathbf{S}$ of the vector field $\mathbf{F} = y\mathbf{i} - x\mathbf{j} + z^3\mathbf{k}$ through the surface Σ in $\mathbb{R}^3$ which is oriented with an upward normal vector and described by

$$(u \cos v, u \sin v, v) \qquad 0 \le u \le 2 \quad 0 \le v \le 2\pi.$$

6. Find the flux of the vector field $\mathbf{F} = x^3\mathbf{i} + y^3\mathbf{j} + z^3\mathbf{k}$ out of the unit sphere in $\mathbb{R}^3$.

7. Find the integral $\int_\gamma \mathbf{F}.d\mathbf{s}$ where $\mathbf{F} = x\mathbf{i} + y^2\mathbf{j} + z^3\mathbf{k}$ and γ is the oriented curve given by

$$(\sin^2 t, \cos^3 t, \sin^4 t) \qquad 0 \le t \le 2\pi.$$

8. One of the two vector fields

$$\mathbf{F} = y^2\mathbf{i} - z^2\mathbf{j} + x^2\mathbf{k}$$
$$\mathbf{G} = (x^3 - 3xy^2)\mathbf{i} + (y^3 - 3x^2y)\mathbf{j} + z\mathbf{k}$$

is conservative, and the other is not. Which is which? Find a potential for the conservative one.

Vector Calculus 20E, Spring 2013, Lecture A, Final exam

Three hours, eight problems. No calculators allowed.

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Simplify answers if you can, but don't worry if you can't!

1. Let γ be the closed curve given by the equations $x = t^2 - t,\ y = 2t^3 - 3t^2 + t$ for $0 \le t \le 1$. Using Green's theorem, find the area enclosed by the curve γ.

2. Find the integral $\int_\gamma \mathbf{F}.\mathbf{ds}$ where $\mathbf{F} = y\mathbf{i} + x\mathbf{j} + z\mathbf{k}$ and γ is the helical curve $x = 2\cos t,\ y = 2\sin t,\ z = t$ for $0 \le t \le 2\pi$, oriented in the direction of increasing t.

3. Find the integral $\int_\Sigma y^2\, \mathrm{dA}$, where Σ is the part of the cylinder $x^2 + y^2 = 4$ lying between the planes $z = 0$ and $z = x + 3$.

4. Let D be the standard unit disc in the xy-plane, and let Σ be the part of the graph of the function $z = xy$ lying over the domain D. Find the surface area of Σ.

5. Let Σ be the hemisphere $x^2 + y^2 + z^2 = 16, z \ge 0$, oriented with the upward normal, and let Let $\mathbf{F}$ be the vector field $(x^2 + z)\mathbf{i} + 3xyz\mathbf{j} + (2xz)\mathbf{k}$. Compute the integral $\int_\Sigma (\nabla \times \mathbf{F}).\mathbf{dA}$

6. Find the flux of the vector field $\mathbf{F} = x^2y\mathbf{i} + z^8\mathbf{j} - 2xyz\mathbf{k}$ out of the surface of the standard unit cube $(0 \le x, y, z \le 1)$ in $\mathbb{R}^3$.

7. Find the integral $\int_\gamma \mathbf{F}.\mathbf{ds}$ where $\mathbf{F} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ and γ is the oriented curve given by

$$(\sin^2 t \cos t, \cos^2 t \sin t, (t - \pi)^4) \qquad 0 \le t \le 2\pi.$$

8. One of the two vector fields

$$\mathbf{F} = 3x^2y\mathbf{i} + x^3\mathbf{j} + 5\mathbf{k}$$
$$\mathbf{G} = (x + z)\mathbf{i} + (z - y)\mathbf{j} + (x - y)\mathbf{k}$$

is conservative, and the other is not. Which is which? Find a potential for the conservative one.

Vector Calculus 20E, Fall 2014, Lecture A, Final Exam

Three hours, eight problems. No calculators allowed.

Please start each problem on a new page.

You will get full credit only if you show all your work clearly.

Simplify answers if you can, but don't worry if you can't!

1. Let D be the upper half of the unit disc, given by $x^2 + y^2 \leq 1, y \geq 0$. Find the average of the function $f(x, y) = y$ over D.

2. Let D^* be the right-hand half of the unit disc, given by $x^2 + y^2 \leq 1, x \geq 0$. Let $D = T(D^*)$, where T is the map $(u, v) \mapsto (u^2 - v^2, 2uv)$. Calculate the area of D.

3. Let C be the curve in the plane described by $t \mapsto (\cos^3 t, \sin t)$ for $0 \leq t \leq 2\pi$. Use Green's theorem to compute the area enclosed by C.

4. Let Σ be the part of the cone $z = \sqrt{x^2 + y^2}$ lying above the standard unit square $0 \leq x, y \leq 1$. Compute the surface area of Σ.

5. Let C be the oriented triangular path formed by travelling from $(1, 0, 0)$ to $(0, 1, 0)$ to $(0, 0, 1)$ and then back to $(1, 0, 0)$ along straight line segments. Let $\mathbf{F}$ be the vector field given by $\mathbf{F}(x, y, z) = (y, x, x^2)$. Compute the circulation of $\mathbf{F}$ around C:

$$\int_C \mathbf{F}.\mathrm{d}\mathbf{s}$$

6. Let γ be the oriented path $t \mapsto (\sqrt{1 + t^2}, \sqrt[3]{1 + t^3}, \sqrt[4]{1 + t^4})$ for $0 \leq t \leq 1$. Let $\mathbf{F}$ be the vector field given by $\mathbf{F}(x, y, z) = (yz, xz, xy)$. Is $\mathbf{F}$ conservative? Calculate

$$\int_\gamma \mathbf{F}.\mathrm{d}\mathbf{s}$$

7. Let Σ be the part of the unit sphere $x^2 + y^2 + z^2 = 1$ with $x, y, z \geq 0$, oriented outwards from the origin as usual. Let $\mathbf{F}$ be the vector field given by $\mathbf{F}(x, y, z) = (y, -x, 1)$. Compute the flux of $\mathbf{F}$ out of Σ:

$$\int_\Sigma \mathbf{F}.\mathrm{d}\mathbf{S}$$

8. Let Σ be the surface made by gluing the upper unit hemisphere (given by $x^2+y^2+z^2 = 1, z \geq 0$) onto the unit disc in the xy-plane (given by $x^2 + y^2 \leq 1, z = 0$); orient the whole surface outwards. Let $\mathbf{F}$ be the vector field given by $\mathbf{F}(x, y, z) = (x^2, xz, 3z)$. Compute the flux of $\mathbf{F}$ out of Σ:

$$\int_\Sigma \mathbf{F}.\mathrm{d}\mathbf{S}$$

Vector Calculus 20E, Winter 2016, Lecture A, Final Exam

Three hours, eight problems. No calculators allowed.

Please start each problem on a new page.

You will get full credit only if you show all your work clearly.

Simplify answers if you can, but don't worry if you can't!

1. Let R be the solid region between the spheres of radius 1 and 2, centred on the origin. Compute

$$\int_R (x+y)^2 \, \mathrm{d}V.$$

2. Let Σ be the surface which is given by $z = 1 - x^2 - y^2, z \geq 0$. Compute the surface area of Σ.

3. Let Σ be the part of the unit sphere $x^2 + y^2 + z^2 = 1$ which lies above the plane $z = \frac{1}{2}$. Compute the average of the function $f(x, y, z) = z$ over Σ.

4. Let γ be the oriented curve parametrised by $t \mapsto (t, t^2, t^3)$ for $0 \leq t \leq 1$ and let $\mathbf{F}$ be the vector field given by $\mathbf{F}(x, y, z) = (x + z, y^3, 1 - x)$. Compute

$$\int_\gamma \mathbf{F} \cdot \mathrm{d}\mathbf{s}.$$

5. Let R be the region $x \geq 0, y \geq 0, z \geq 0, x+y+z \leq 1$, and let Σ be its boundary surface, oriented outwards. Let $\mathbf{F}$ be the vector field given by $\mathbf{F}(x, y, z) = (4x - z^2, x + 3z, 6 - z)$. Compute

$$\int_\Sigma \mathbf{F} \cdot \mathrm{d}\mathbf{A}.$$

6. Let Σ be the part of the cone given by $z = \sqrt{x^2 + y^2}, 0 \leq z \leq 1$, oriented outwards. Let $\mathbf{F}$ be the vector field given by $\mathbf{F}(x, y, z) = (-z^2 y, z^2 x, z^4)$. Compute

$$\int_\Sigma (\nabla \times \mathbf{F}) \cdot \mathrm{d}\mathbf{A}.$$

7. Let Σ be the unit hemisphere $x^2 + y^2 + z^2 = 1, x \geq 0$, oriented using the outward normal, and let $\mathbf{F}$ be the vector field given by $\mathbf{F}(x, y, z) = (y, x, z)$. Compute the flux

$$\int_\Sigma \mathbf{F} \cdot \mathrm{d}\mathbf{A}.$$

8. Find the potential function ϕ which satisfies $\phi(\frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}) = 0$ and $\nabla\phi = \mathbf{F}$, where $\mathbf{F}$ is the vector field given by $\mathbf{F}(x, y, z) = (\sin y - z \cos x, x \cos y + \sin z, y \cos z - \sin x)$.

Vector Calculus 20E, Winter 2017, Lecture B, Final Exam

Three hours, eight problems. No calculators allowed.

Please start each problem on a new page.

You will get full credit only if you show all your work clearly.

Simplify answers if you can, but don't worry if you can't!

1. Let Σ be the part of the plane $2x + y + z = 6$ where $x, y, z \geq 0$. Compute

$$\int_\Sigma (x+z)\ \mathrm{d}A.$$

2. Let Σ be the piece of a cylinder given by $x^2 + y^2 = 4$, $y \geq 0$ and $0 \leq z \leq 1$, oriented with the outward normal. Let $\mathbf{F}$ be the vector field given by $\mathbf{F}(x, y, z) = (x, 1, z^2)$. Compute

$$\int_\Sigma \mathbf{F} \cdot \mathrm{d}\mathbf{A}.$$

3. Let Σ be the hemisphere given by $(x-1)^2 + y^2 + z^2 = 1$ and $z \geq 0$, oriented with the upward normal. Let $\mathbf{F}$ be the vector field given by $\mathbf{F}(x, y, z) = (-e^{\sin z} y, x + \sin z, \cos x)$. Compute

$$\int_\Sigma (\nabla \times \mathbf{F}) \cdot \mathrm{d}\mathbf{A}.$$

4. Let R be the solid pipe defined by $1 \leq x^2 + y^2 \leq 4$ and $0 \leq z \leq 10$, and let Σ be its boundary surface, oriented out of R. Let $\mathbf{F}$ be the vector field given by $\mathbf{F}(x, y, z) = (x + yz, y + x^2 z^3, xyz)$. Compute

$$\int_\Sigma \mathbf{F} \cdot \mathrm{d}\mathbf{A}.$$

5. Let R be the part of the unit ball given by $x^2 + y^2 + z^2 \leq 1$, $x, y, z \geq 0$. Find the average, over R, of the distance from the origin.

6. Let γ be some curve which runs from $(0, 0, 0)$ to $(1, 2, 3)$. Let $\mathbf{F}$ and $\mathbf{G}$ be vector fields given by $\mathbf{F}(x, y, z) = (3x^2y^2z, 2x^3yz, x^3y^2)$ and $\mathbf{G}(x, y, z) = (3x^2y^2z, 2x^3yz, x^2y^3$. Evaluate, or say why it can't be evaluated without further information, the integrals

$$\int_\gamma \mathbf{F} \cdot \mathrm{d}\mathbf{s} \qquad \int_\gamma \mathbf{G} \cdot \mathrm{d}\mathbf{s}.$$

7. Let γ be the straight-line segment running from $(1, 2, 3)$ to $(4, 0, 2)$. Let $\mathbf{F}$ be the vector field given by $\mathbf{F}(x, y, z) = (xz, 3y, 1)$. Calculate

$$\int_\gamma \mathbf{F} \cdot \mathrm{d}\mathbf{s}.$$

8. Let D be the parallelogram whose vertices are $(0, 1), (1, 0), (2, 2), (3, 1)$. Evaluate

$$\int_D \cos(x+y)\ \mathrm{d}A.$$