2012 Final

DISCLAIMER: Answers not guaranteed to be correct. Study at your own risk!

① Choosing the parametrization

$\gamma(t) = \begin{bmatrix} 2\cos(t) \\ \sin(t) \end{bmatrix}, t \in [0, 2\pi]$

(To see why, note that $x^2 + 4y^2 = 4 \longleftrightarrow \frac{1}{4}x^2 + y^2 = 1$
$\longleftrightarrow (\frac{1}{2}x)^2 + y^2 = 1$.
So we want
$\frac{1}{2}x(t) := \cos(t)$
$y(t) := \sin(t)$)

we get

$\int_\gamma (4y - 3x)dx + (x - 4y)dy$

$dx \longmapsto \frac{\partial x}{\partial t} = -2\sin(t)dt$

$dy \longmapsto \partial y/\partial t = \cos(t)dt$

$= \int_0^{2\pi} (4(\sin(t)) - 3(2\cos(t)))(-2\sin(t)) + (2\cos(t) - 4\sin(t))(\cos(t))\,dt$

$= \int_0^{2\pi} -8\sin^2(t) + 12\sin(t)\cos(t) + 2\cos^2(t) - 4\sin(t)\cos(t)\,dt$

$= \int_0^{2\pi} 2\sin^2(t) + 2\cos^2(t) - 10\sin^2(t) + 8\sin(t)\cos(t)\,dt$

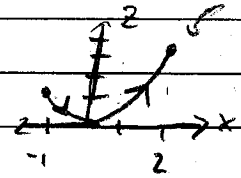
$= \int_0^{2\pi} 2 - \underbrace{10\sin^2(t)} + \underbrace{8\sin(t)\cos(t)\,dt}_{\text{use u-sub}}$

4π

$\int_0^{2\pi} -10\sin^2(t)dt$

$= -10\int_0^{2\pi} \left(\frac{1-\cos(2t)}{2}\right)dt$

Putting it all together, we get -6π.

(2) 

γ (in x-z plane)

Choosing the parametrization

$$\gamma(t) = \begin{bmatrix} t \\ 0 \\ t^2 \end{bmatrix}, \quad t \in [-1, 2], \quad \Longrightarrow \quad \gamma'(t) = \begin{bmatrix} 1 \\ 0 \\ 2t \end{bmatrix}$$

$$F(\gamma(t)) = \begin{bmatrix} 0 \\ t \\ t^2 \end{bmatrix}$$

So
$$\int_\gamma F \, d\vec{s} = \int_{-1}^{2} F(\gamma(t)) \cdot \gamma'(t)\, dt = \int_{-1}^{2} 2t^3 \, dt$$
$$= \tfrac{1}{2} t^4 \Big|_{t=-1}^{2} = 8 - \tfrac{1}{2}(1)$$
$$= 15/2$$

OR

Notice that if $f(x,y,z) = xy + \frac{1}{2}z^2$, we get $\nabla f = \begin{bmatrix} y \\ x \\ z \end{bmatrix} = F$,

So by Fundamental Theorem of Calculus,
$$\int_\gamma F \cdot d\vec{s} = \int_{t=-1}^{2} F(\gamma(t)) \cdot \gamma'(t)\, dt$$
$$= \int_{t=-1}^{2} \tfrac{d}{dt}\left(f(\gamma(t))\right) = f(\gamma(2)) - f(\gamma(-1))$$
$$= f(2,0,4) - f(-1,0,1)$$
$$= 16/2 - 1/2 = 15/2.$$

(3) R is a subset of a plane, which can be thought of as the span of 2 of the edges on R. Let's choose the edges

$$\vec{v}_1 = \left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad \& \quad \vec{v}_2 = \left(\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$$

$$\vec{n} = \vec{v}_1 \times \vec{v}_2 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 0 & 0 \\ -1 & 1 & 1 \end{vmatrix} = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$$

Our plane that contains R is given by

$$\begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} x-0 \\ y-0 \\ z-0 \end{bmatrix} = 0 \quad \Rightarrow \quad z = y$$

$\uparrow$ ($\vec{0} \in$ plane)

So a natural parametrization is

$$\Phi(x,y) = \begin{bmatrix} x \\ y \\ y \end{bmatrix}, \quad \text{where } x \in [0,1], \ y \in [0,1]$$

Hence

$$\int_R xyz \, dS = \int_0^1 \int_0^1 xy(y) \left(\left\| \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} \right\| \right) dx\,dy$$

$$= \int_0^1 \int_0^1 xy^2 (\sqrt{2})\, dx\,dy$$

$$= \sqrt{2}\left(\tfrac{1}{6}\right)$$

(4) Recall that

$$\text{Area}(\Sigma) = \iint_\Sigma 1 \cdot d\vec{S} = \int_{v_0}^{v_1} \int_{u_0}^{u_1} \|T_u \times T_v\| \, du\,dv$$

Here, we have

$$\Phi(u,v) = \begin{bmatrix} u\cos(v) \\ u\sin(v) \\ u^2 \end{bmatrix}, \quad 0 \le u \le 2, \ 0 \le v \le 2\pi$$

$$T_u = \begin{bmatrix} \cos(v) \\ \sin(v) \\ 2u \end{bmatrix}, \quad T_v = \begin{bmatrix} -u\sin(v) \\ u\cos(v) \\ 0 \end{bmatrix}$$

$$T_u \times T_v = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \cos(v) & \sin(v) & 2u \\ -u\sin(v) & u\cos(v) & 0 \end{vmatrix} = \begin{bmatrix} -2u^2\cos(v) \\ 2u^2\sin(v) \\ u \end{bmatrix}$$

So

$$\|T_u \times T_v\| = \sqrt{4u^4\cos^2(v) + 4u^4\sin^2(v) + u^2}$$
$$= \sqrt{4u^4 + u^2}$$
$$= u\sqrt{4u^2+1}$$

Hence

$$\text{Area}(\Sigma) = \int_0^{2\pi}\int_0^2 u\sqrt{4u^2+1}\,du\,dv$$

Letting $\alpha := 4u^2+1 \Rightarrow \alpha \in [1, 17]$

$d\alpha = 8u\,du$

$\Rightarrow u\,du = 1/8\,d\alpha$

So $\text{Area}(\Sigma) = 2\pi(1/8)\int_0^2 \sqrt{\alpha}\,d\alpha$

$$= \pi/4 \; 2/3 (\alpha)^{3/2}\Big|_{\alpha=1}^{17} = \pi/6\,(17\sqrt{17} - 1).$$

⑤ Here, $\Phi(u,v) = \begin{bmatrix} u\cos(v) \\ u\sin(v) \\ v \end{bmatrix}$, $u \in [0,2]$, $v \in [0, 2\pi]$

$T_u = \begin{bmatrix} \cos(v) \\ \sin(v) \\ 0 \end{bmatrix}$, $T_v = \begin{bmatrix} -u\sin(v) \\ u\cos(v) \\ 1 \end{bmatrix}$

$\Rightarrow T_u \times T_v = \begin{vmatrix} i & j & k \\ \cos(v) & \sin(v) & 0 \\ -u\sin(v) & u\cos(v) & 1 \end{vmatrix} = \begin{bmatrix} \sin(v) \\ -\cos(v) \\ u \end{bmatrix}$

note that since $u \geq 0$, this is the upward facing normal, which is the one we want

$\Rightarrow F(\Phi(u,v)) \cdot (T_u \times T_v) = \begin{bmatrix} u\sin(v) \\ -u\cos(v) \\ v^3 \end{bmatrix} \cdot \begin{bmatrix} \sin(v) \\ -\cos(v) \\ u \end{bmatrix}$

$= u + uv^3$

Hence

$$\iint_{\Sigma} F \cdot d\vec{S} = \int_0^{2\pi}\int_0^{2} u + uv^3 \, du\, dv$$

$$= \int_0^{2\pi} u^2/2 + u^2/2 \, v^3 \Big|_{u=0}^{2}$$

$$= \int_0^{2\pi} 2 + 2v^3 \, dv$$

$$= 2v + \tfrac{1}{2} v^4 \Big|_{v=0}^{2\pi} = 4\pi + \tfrac{1}{2}(16\pi^4)$$

$$= 4\pi + 8\pi^4$$

(6) By Gauss' Theorem,

$$\iiint_B \operatorname{div}(F)\, dV = \iint_{\Sigma} F\, d\vec{S}$$

where B is the solid unit ball, & Σ is the hollow unit sphere.

$$\operatorname{div}(F) = 3x^2 + 3y^2 + 3z^2$$

Using spherical coordinates,

fudge factor

$$\iiint_B 3x^2 + 3y^2 + 3z^2 \, dV = 3 \int_0^{2\pi}\int_0^{1}\int_0^{\pi/2} \rho^2 (\cos^2\theta \sin^2\varphi + \sin^2\theta \sin^2\varphi + \cos^2\varphi)\, \rho^2 \sin^2\varphi \, d\varphi$$

$$= 3 \int_0^{2\pi}\int_0^{1}\int_0^{\pi/2} \rho^4 \sin^2\varphi \, d\varphi\, d\rho\, d\theta$$

$$= 6\pi \int_0^1 \rho^4 \int_0^{\pi/2} \tfrac{1}{2}(1 - \cos(2\varphi))\, d\varphi\, d\rho$$

$$= 3\pi \int_0^1 \rho^4 \left(\varphi - \tfrac{1}{2}\sin(2\varphi) \Big|_{\varphi=0}^{\pi/2} \right) d\rho$$

$= 3\pi \int_0^1 \rho^4 (\pi/2)\, d\rho$

$= \frac{3}{10} \pi^2$

(7) This looks like a crazy path.... wouldn't it be nice if we could chose a different path? We would need either:

① $F = \nabla f$, for some $f: \mathbb{R}^3 \to \mathbb{R}$

or ② $\nabla \times F = 0$

Note that $f(x,y,z) = \frac{1}{2}x^2 + \frac{1}{3}y^3 + \frac{1}{4}z^4$ satisfies $\nabla f = F$, so we can choose any path to integrate over, so long as the endpoints are the same. Well,

$$\gamma(0) = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \qquad \gamma(2\pi) = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

How nice! Since the endpoints are the same, we get $\int_\gamma F \cdot d\vec{s} = 0$.

(8) Recall there are 2 useful ways to check if a vector field is conservative

① $F = \nabla f$, for some $f: \mathbb{R}^3 \to \mathbb{R}$

or ② $\nabla \times F = 0$.

We czech:

$$\nabla \times F = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ y^2 & -z^2 & x^2 \end{vmatrix} = \begin{bmatrix} 2z \\ -2x \\ 2y \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \text{ everywhere}$$

So F is not conservative, meaning G must be of the form $G = \nabla f$ for some f. That is

$$G(x,y,z) = \begin{bmatrix} x^3 - 3xy^2 \\ y^3 - 3x^2y \\ z \end{bmatrix} = \begin{bmatrix} \partial f/\partial x \\ \partial f/\partial y \\ \partial f/\partial z \end{bmatrix}$$

$\int \partial f/\partial x \, dx = f(x,y,z) - g(y,z)$, for some $g(y,z)$.

$\Rightarrow \int x^3 - 3xy^2 dx = 1/4 x^4 - 3/2 x^2y^2 + g(y,z) = f(x,y,z)$

We also need

① $\partial/\partial y \left(1/4 x^4 - 3/2 x^2y^2 + g(y,z)\right) = y^3 - 3x^2y$

$$\begin{gathered} \| \\ -3x^2y + \partial g/\partial y = y^3 - 3x^2y \end{gathered}$$

$\Rightarrow \partial g/\partial y = y^3$

② $\partial/\partial z \left(1/4 x^4 - 3/2 x^2y^2 + g(y,z)\right) = z$

$$\begin{gathered} \| \\ \partial g/\partial z = z \end{gathered}$$

A g which satisfies this is

$$g(y,z) = 1/4 y^4 + 1/2 z^2$$

So we guess that

$$f(x,y,z) = 1/4 x^4 - 3/2 x^2y^2 + 1/4 y^4 + 1/2 z^2$$

<u>Check</u>: $\nabla f = \begin{bmatrix} x^3 - 3xy^2 \\ y^3 - 3x^2y \\ z \end{bmatrix} = F$

So we win.

Spring 2013, Final exam

Answers/Solutions not guaranteed!!

1) Let γ be closed curve given by $x=t^2-t,\ y=2t^3-3t^2+t$ $0\le t\le 1$.
Use greens to find area of enclosed curve

Stokes/Greens: $\int_\gamma \bar{F}\cdot d\bar{s} = \iint_S (\nabla\times F)\cdot d\bar{S}$ where $\partial S=\gamma$

Let $F= -y\,dx + x\,dy$

Then $\nabla\times F = \left(\frac{\partial x}{\partial x} - \frac{\partial(y)}{\partial y}\right)\hat{z} = 2\hat{z}$

So $\int_\gamma \bar{F}\cdot d\bar{s} = 2A(S)$ if S is area bounded by γ in plane.

Thus $A(S) = \frac{1}{2}\int_\gamma \bar{F}\cdot d\bar{s}$

$$= \frac{1}{2}\int_0^1 dt\left(F(c(t))\cdot c'(t)\right)$$

$$= \frac{1}{2}\int_0^1 dt \left(-(2t^3-3t^2+t),\ t^2-t\right)\cdot\left(2t-1,\ 6t^2-6t+1\right)$$

$$= \frac{1}{2}\int_0^1 dt\left(-(2t^3-3t^2+t)(2t-1) + (t^2-t)(6t^2-6t+1)\right)$$

$$= \frac{1}{2}\int_0^1 dt\Big(-4t^4+2t^3+6t^3-3t^2-2t^2+t+6t^4-6t^3+t^2$$
$$-6t^3+6t^2-t\Big)$$

$$= \frac{1}{2}\int_0^1 dt\left(2t^4-4t^3+2t^2\right) = \frac{1}{2}\left(\frac{2t^5}{5}-\frac{4t^4}{4}+\frac{2t^3}{3}\Big|_0^1\right)$$

$$= \frac{1}{2}\left(\frac{2}{5} - 1 + \frac{2}{3}\right) = \frac{1}{2}\left(\frac{6}{15} - \frac{15}{15} + \frac{10}{15}\right) = \boxed{\frac{1}{30}}$$

2) Find $\int_\gamma \bar{F}\cdot d\bar{s}$, $\bar{F} = y\hat{i} + x\hat{j} + z\hat{k}$,

$$\gamma = (2\cos t, 2\sin t, t) \quad t \in [0, 2\pi]$$

$$\int_\gamma \bar{F}\cdot d\bar{s} = \int_0^{2\pi} dt \; F(c(t))\cdot c'(t) = \int_0^{2\pi} dt\,(2\sin t, 2\cos t, t)\cdot(-2\sin t, 2\cos t, 1)$$

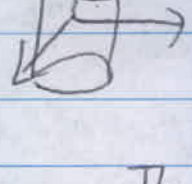
$$= \int_0^{2\pi} \left(-4\sin^2(t) + 4\cos^2(t) + t\right) dt = \int_0^{2\pi}\left(-4\left(\frac{1-\cos(2t)}{2}\right) + 4\left(\frac{1+\cos(2t)}{2}\right) + t\right)$$

$$= \int_0^{2\pi} dt\left(-2 + 2 + 4\cos(2t) + t\right) = \int_0^{2\pi} dt\left(t + 4\cos(2t)\right)$$

$$= \frac{t^2}{2} + 2\sin(2t)\Big|_0^{2\pi} = \boxed{\frac{(2\pi)^2}{2}}$$

3) Find $\int_\Sigma y^2\, dA$, Σ part of cylinder $x^2+y^2=4$ b/w $z=0$

and $z = x+3$



Let $\Phi(\theta, z) = (2\cos(\theta), 2\sin(\theta), z)$ for

$$0 \le \theta \le 2\pi, \quad 0 \le z \le 2\cos\theta + 3$$

Then $\int_\Sigma y^2 dA = \int_0^{2\pi} d\theta \int_0^{2\cos\theta+3} dz \; (2\sin(\theta))^2 \, \|T_\theta \times T_z\| \cdot$

$$\|T_\theta \times T_z\| = \left\| \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ -2\sin\theta & 2\cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix} \right\| = \|2\cos\theta\, \hat{x} + 2\sin\theta\, \hat{y}\|$$

$$= \left(4\cos^2 + 4\sin^2\right)^{1/2} = 2$$

Thus $I = \int_0^{2\pi} d\theta \int_0^{2\cos\theta+3} dz \left(2^2 \sin^2\theta\right) 2$

$$= \int_0^{2\pi} d\theta \quad 8\sin^2\theta \, z \Big|_0^{2\cos\theta+3}$$

$$= \int_0^{2\pi} d\theta \quad 8\sin^2\theta \left(2\cos\theta+3\right)$$

$$= \int_0^{2\pi} d\theta \left(16\sin^2\theta\cos\theta + 24\sin^2\theta\right) = \int_0^{2\pi} d\theta \left(16\sin^2\theta\cos\theta + 24\left(\frac{1-\cos(2\theta)}{2}\right)\right)$$

$$= \frac{16}{3}\sin^3\theta + 12\theta - 6\sin(2\theta) \Big|_0^{2\pi}$$

$$= 12(2\pi) = \boxed{24\pi}$$

4) Let D be unit disc in xy plane, Σ be part of graph of $z=xy$ over D. Find surface area of Σ

Let $\Phi(x,y) = (x, y, xy)$ for $\{x^2+y^2 \le 1\} = D$

Then $A(\Sigma) = \iint_D \|T_x \times T_y\| \, dA$

$$\|T_x \times T_y\| = \left\| \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ 1 & 0 & y \\ 0 & 1 & x \end{vmatrix} \right\| = \| -y\hat{x} - x\hat{y} + \hat{z} \| = (y^2 + x^2 + 1)^{1/2}$$

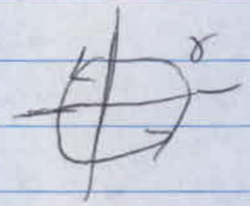
$$\text{Thus } A = \iint_D (y^2+x^2+1)^{1/2}\, dA$$

$$= \int_0^{2\pi} d\theta \int_0^1 dr\, (r^2+1)^{1/2}\, r$$

$$= (2\pi)\ \frac{1}{2}\frac{2}{3}(r^2+1)^{3/2}\Big|_0^1 = \boxed{\frac{2\pi}{3}\left(2^{3/2}-1\right)}$$

5) Let Σ be $x^2+y^2+z^2=16$, $z \geq 0$, oriented w/ upward normal.
Let $\bar{F} = (x^2+z)\hat{i} + 3xyz\,\hat{j} + (2xz)\hat{k}$
Compute $\int_\Sigma (\nabla \times F)\cdot d\bar{A} \equiv I$

Stokes $\Rightarrow I = \int_\gamma \bar{F}\cdot d\bar{s}$ where γ is circle of radius 4 in xy plane traversed ccw viewed from above.



$\gamma(t) = (4\cos t, 4\sin t, 0)\quad t \in [0, 2\pi]$

$$\text{So } I = \int_0^{2\pi} (16\cos^2 t, 0, 0)\cdot(-4\sin(t), 4\cos t, 0)\, dt$$

$$= \int_0^{2\pi} -64\cos^2 t \sin t\, dt = +\frac{64}{3}\cos^3(t)\Big|_0^{2\pi} \boxed{=0}$$

6) Find flux of $F = x^2y\,\hat{i} + z^8\,\hat{j} - 2xyz\,\hat{k}$ out of surface of std. unit cube $(0 \le x, y, z \le 1)$ in $\mathbb{R}^3$

Divergence Thm $\quad \iiint_V (\nabla\cdot F)\,dV = \iint_S \bar{F}\cdot d\bar{S} \equiv I$

$$\nabla\cdot F = 2xy + 0 - 2xy = 0$$

$$\boxed{\text{so } I = 0}$$

7) Find $\int_\gamma \bar{F}\cdot d\bar{s}$ w/ $F = x\hat{i} + y\hat{j} + z\hat{k}$ and γ is oriented curve given by $\gamma = (\sin^2 t\cos t,\ \cos^2 t\sin t,\ (t-\pi)^4)$ $t \in [0, 2\pi]$

method i) use stokes, γ is closed, so look for S s.t. $\partial S = \gamma$

Compute $\nabla\times F = 0$, $\boxed{\text{so } I = 0}$

ii) $\int_\gamma \bar{F}\cdot d\bar{s} = \int_0^{2\pi} F(\gamma(t))\cdot\gamma'(t)\,dt = \dots$

8) $\bar{F} = 3x^2y\,\hat{i} + x^3\,\hat{j} + 5\hat{k}$

$\bar{G} = (x+z)\,\hat{i} + (z-y)\,\hat{j} + (x-y)\,\hat{k}$

$$\nabla\times G = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial_x & \partial_y & \partial_z \\ x+z & z-y & x-y \end{vmatrix} = \hat{x}(-1-1) - \hat{y}(1-1) + \hat{z}(0) \neq 0$$

Not conservative

$\nabla \times F = 0$, Conservative

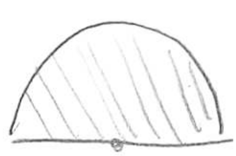
$F = \nabla f$; $\quad \frac{\partial f}{\partial x} = 3x^2 y \Rightarrow f(x,y,z) = x^3 y + g(y,z)$

$$\frac{\partial f}{\partial y} = x^3 \Rightarrow \frac{\partial g}{\partial y} = 0 \Rightarrow g = g(z)$$

$$\frac{\partial f}{\partial z} = 5 \Rightarrow \frac{\partial g}{\partial z} = 5 \Rightarrow g = 5z + C$$

So $f = x^3 y + 5z + C$

#1



D

$$\text{Average} = \frac{\iint_D f\,dA}{\iint_D dA}$$

$$= \frac{\iint_D y\,dx\,dy}{\text{Area}(D)} = \frac{\int_0^{\pi}\int_0^{1} r^2\sin\theta\,dr\,d\theta}{\pi/2} = \frac{4}{3\pi} \qquad \#$$

#2

D^*

$$T(u,v) = (u^2-v^2,\ 2uv) = (s,t).$$

$$T(D^*) = \iint_{D^*} \left|\frac{\partial(s,t)}{\partial(u,v)}\right| du\,dv.$$

As $\dfrac{\partial(s,t)}{\partial(u,v)} = \begin{vmatrix} 2u & -2v \\ 2v & 2u \end{vmatrix} = 4u^2+4v^2$, we have

$$T(D^*) = \iint_{D^*} 4(u^2+v^2)\,du\,dv = \int_{-\pi/2}^{\pi/2}\int_0^1 4r^2\cdot r\,dr\,d\theta = \pi \qquad \#$$

#3

By Green's Theorem, Area $= \frac{1}{2}\int_\gamma x\,dy - y\,dx = \int_\gamma x\,dy$.

Here we use the second formula.

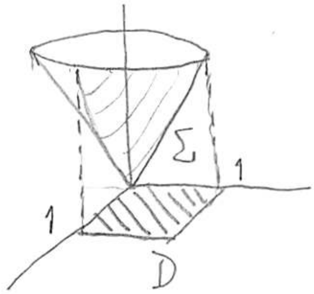
$$\int_0^{2\pi} x\,dy = \int_0^{2\pi} \cos^3 t\,(\cos t)\,dt = \int_0^{2\pi} (\cos^2 t)^2\,dt = \int_0^{2\pi} \left(\frac{1+\cos 2t}{2}\right)^2 dt$$

$$= \frac{1}{4}\int_0^{2\pi} 1 + 2\cos(2t) + \cos^2(2t)\,dt = \frac{1}{4}(2\pi) + 0 + \frac{1}{4}\int_0^{2\pi} \frac{1}{2} + \frac{\cos 4t}{2}\,dt$$

$$= \frac{1}{4}(2\pi) + 0 + \frac{1}{4}\left(\frac{1}{2}(2\pi)\right) + 0 = \frac{3\pi}{4} \qquad \#$$

#4

$$\text{Area } \Sigma = \iint_\Sigma 1\,dS = \iint_D \|T_x \times T_y\|\,dx\,dy$$



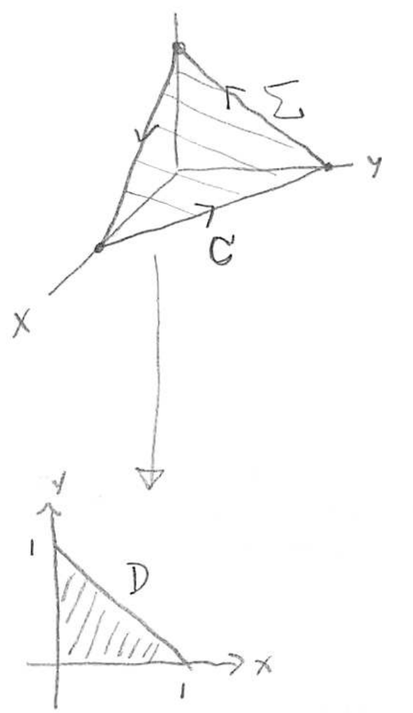
$$= \int_0^1\int_0^1 \|T_x \times T_y\|\,dx\,dy$$

Σ is $\Phi(x,y) = (x, y, z(x,y))$ where $z(x,y) = \sqrt{x^2+y^2}$

Then $T_x \times T_y = \left(-\frac{\partial z}{\partial x}, -\frac{\partial z}{\partial y}, 1\right) = \left(-\frac{x}{z}, -\frac{y}{z}, 1\right)$, so

$\|T_x \times T_y\| = \sqrt{\frac{x^2+y^2+z^2}{z^2}} = \sqrt{\frac{2z^2}{z^2}} = \sqrt{2}$. Hence Area $\Sigma = \sqrt{2}$ #

#5



Stoke's Thm : $\displaystyle\int_C \vec{F}\cdot d\vec{s} = \iint_\Sigma (\nabla\times\vec{F})\cdot d\vec{S}$

$$\nabla\times\vec{F} = \begin{vmatrix} i & j & k \\ \partial_x & \partial_y & \partial_z \\ y & x & x^2 \end{vmatrix} = (0,-2x,0)$$

Σ : $\Phi(x,y) = (x,y,z(x,y))$ where $x+y+z=1$.

Then $T_x \times T_y = (1,1,1)$.

$$\iint_\Sigma (\nabla\times\vec{F})\cdot d\vec{S} = \iint_D (0,-2x,0)\cdot(1,1,1)\,dx\,dy = \int_0^1\int_0^{1-y} -2x\,dx\,dy = -\frac{1}{3} \quad \#$$

#6

Choose $f(x,y,z) = xyz$. Then $\nabla f = F$, so

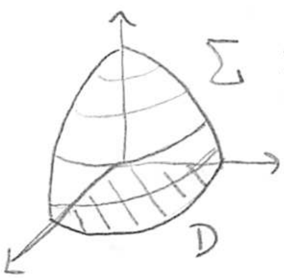
F is conservative. Notice that $\gamma(0) = (1,1,1)$, $\gamma(1) = (2^{1/2}, 2^{1/3}, 2^{1/3})$

By Fundamental Theorem of Line Integral, $\displaystyle\int_\gamma \nabla f\cdot dS = f(\gamma(1)) - f(\gamma(0))$

$$= 2^{13/12} - 1 \quad \#$$

#7

Because the surface is not closed, you can't apply Gauss's Theorem here!



Σ : $\Phi(x,y) = (x, y, z(x,y))$ where $z = \sqrt{1-x^2-y^2}$.

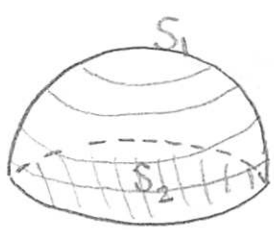
Then $T_x \times T_y = \left(-\frac{\partial z}{\partial x}, -\frac{\partial z}{\partial y}, 1\right) = \left(\frac{x}{z}, \frac{y}{z}, 1\right)$

So $\iint_\Sigma F \cdot dS = \iint_D (y, -x, 1) \cdot \left(\frac{x}{z}, \frac{y}{z}, 1\right) dx\,dy = \iint_D dx\,dy = \text{Area}(D)$

$= \frac{\pi}{4}$ #

#8

Because $\Sigma = S_1 \cup S_2$ is a closed surface, by Gauss's Thm



we have $\iint_\Sigma \vec{F} \cdot d\vec{S} = \iiint_W \nabla \cdot \vec{F}\, dV = \iiint_W (2x+3)\, dV$

Notice that $\int_{-\sqrt{1-y^2-z^2}}^{\sqrt{1-y^2-z^2}} (2x)\, dx = 0$ because $x \mapsto 2x$ is an odd function

Hence, $\iint_\Sigma \vec{F} \cdot d\vec{S} = \iiint_W 3\, dV = 3\,\text{Vol}(W) = \frac{3}{2}\,\text{Vol}(\text{Unit Ball}) = \frac{3}{2}\left(\frac{4}{3}\pi 1^3\right)$

$= 2\pi$ #

Solutions for 20 E Final, 2016

① There are lots of ways to do this: various choices of coord system & symmetry tricks are available.

Most obvious (to me) is to say

$$\int_R (x+y)^2 \, dV = \int_R (x^2 + 2xy + y^2) \, dV$$

By rotational symmetry of R about the origin, the $\int_R xy\,dV = 0$ and $\int_R x^2\,dV = \int_R y^2\,dV \left(= \int_R z^2\,dV \right)$.

Let's do "$\int_R z^2\,dV$" using spherical coords, for example ($dV = r^2 \sin\phi \, dr\,d\theta\,d\phi$)

(this is the easiest of the three, because $z = r\cos\phi$ whereas $x = r\cos\phi\cos\theta$ and $y = r\cos\phi\sin\theta$).

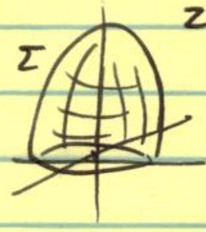
$$\int_R z^2\,dV = \int_1^2 dr \int_0^{2\pi} d\theta \int_0^{\pi} d\phi \, . \underbrace{(r\cos\phi)^2}_{z^2} . \underbrace{r^2 \sin\phi}_{\text{Jacobian factor}}$$

$$= 2\pi . \int_1^2 r^4\,dr . \int_0^{\pi} \cos^2\phi \sin\phi \, d\phi$$

$$= 2\pi . \left(\frac{2^5 - 1^5}{5} \right) . \left[-\frac{1}{3} \cos^3\phi \right]_0^{\pi}$$

$$= 2\pi . \frac{31}{5} . \frac{2}{3} = \frac{124\pi}{15}$$

Therefore $\int_R (x+y)^2\,dV = 2 . \int_R z^2\,dV = \dfrac{248\pi}{15}$

2 

$z = 1 - x^2 - y^2$

$z \geq 0$

Use 'graph' parametrization

$z = g(u,v)$

$(u,v) \in$ unit disc

so surface area is $\displaystyle\iint_{\text{unit disc}} du\,dv \,.\, \sqrt{1 + g_u^2 + g_v^2}$

$$= \iint_{\text{unit disc}} du\,dv \sqrt{1 + 4u^2 + 4v^2}$$

change to polar coords $\quad du\,dv = r\,dr\,d\theta \quad , \; u^2 + v^2 = r^2$

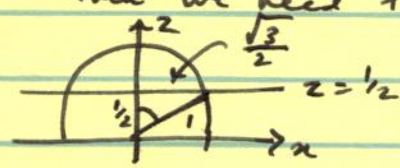
$$= \int_0^1 dr \int_0^{2\pi} d\theta \,.\, r \,.\, \sqrt{1 + 4r^2}$$

$$= 2\pi \,.\, \left[\frac{2}{3} \frac{(1 + 4r^2)^{3/2}}{8} \right]_0^1$$

$$= \cancel{2\pi}\; \underline{\underline{\frac{\pi}{6}\left(5^{3/2} - 1\right)}}$$

3 

If we use spherical polar coords then we need the ϕ limits : by trigonometry



the angle is actually $60^\circ = \cancel{\pi/3}\; \pi/3$

so we'd compute the average of z as

$$\int_0^{2\pi} d\theta \int_0^{\pi/3} d\phi \,.\, \underset{\substack{\uparrow \\ \text{Jacobian} \\ \text{factor}}}{\sin\phi} \,.\, \underset{\substack{\uparrow \\ \text{"z"}}}{\cos\phi} \Bigg/ \underbrace{\int_0^{2\pi} d\theta \int_0^{\pi/3} d\phi \,.\, \sin\phi \,.\, 1}_{\text{gets area of the cap.}}$$

now $\displaystyle\int_0^{\pi/3} d\phi \sin\phi\cos\phi = \int_0^{\pi/3} \frac{1}{2}\sin 2\phi \, d\phi = \left[-\frac{1}{4}\cos 2\phi \right]_0^{\pi/3}$

$$= -\frac{1}{4}\cos\frac{2\pi}{3} + \frac{1}{4}$$

$$= -\frac{1}{4}\left(-\frac{1}{2}\right) + \frac{1}{4} = 3/8$$

and $\displaystyle\int_0^{\pi/6} \sin\phi \, d\phi = \Big[-\cos\phi \Big]_0^{\pi/3} = -\tfrac{1}{2} - (-1) = \tfrac{1}{2}$

The '2π's cancel & so average is $\underline{\underline{3/4}}$.

(It's actually easier if you use cylindrical coordinates (z, θ) because then $dA = dz\, d\theta$ on the unit sphere, & we get more directly

$$\int_{\frac{1}{2}}^{1} dz \int_0^{2\pi} d\theta \cdot z \Big/ \int_{\frac{1}{2}}^{1} dz \cdot \int_0^{2\pi} d\theta \cdot 1 = \underline{\underline{\tfrac{3}{4}}}. \Big]$$

(I wouldn't recommend the 'graph' parametrisation in this case — too many $\dfrac{1}{\sqrt{1-x^2-y^2}}$ factors! — but it will of course still work, it's just messier.]

(4) $\underline{s}(t) = (t, t^2, t^3) \quad 0 \le t \le 1 \qquad \dfrac{d\underline{s}}{dt} = (1, 2t, 3t^2)$

$$F(\underline{s}(t)) = \begin{pmatrix} t + t^3 \\ t^6 \\ 1 - t \end{pmatrix}$$

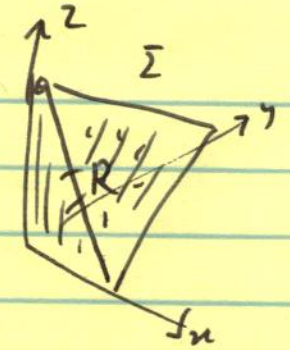
so $$\int_\gamma \underline{F} \cdot d\underline{s} = \int_0^1 \begin{pmatrix} t + t^3 \\ t^6 \\ 1-t \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 2t \\ 3t^2 \end{pmatrix} dt$$

$$= \int_0^1 \left(t \cancel{+t^3} + 2t^7 + 3t^2 - 2t^3 \right) dt$$

$$= \frac{1}{2} + \frac{2}{8} + \frac{3}{3} - \frac{2}{4}$$

$$= \underline{\underline{5/4}}$$

5



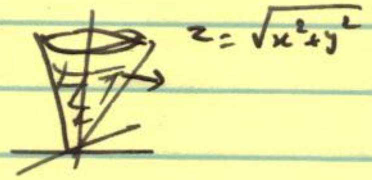
Because the boundary surface has 4 separate parts, this will probably be easiest if we use the divergence thm:

$\nabla \cdot \underline{F} = 4 + 0 - 1 = 3$!

$$\therefore \int_\Sigma \underline{F} \cdot d\underline{A} = \int_R (\nabla \cdot \underline{F})\, dV = 3 \cdot \mathrm{vol}\, R = 3 \cdot \frac{1}{6} = \frac{1}{2}$$

$[\frac{1}{3} \cdot \text{base} \times \text{height}]$

6



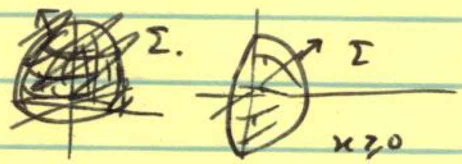
Use Stokes' thm (because the boundary curve of Σ is a nice circle parametrised by $\theta \longmapsto (\cos\theta, \sin\theta, 1)$ $0 \le \theta \le 2\pi$

$$\int_\Sigma (\nabla \times F) \cdot d\underline{A} = \int_{\partial\Sigma} \underline{F} \cdot d\underline{s} = \int_0^{2\pi} \begin{pmatrix} -y \\ x \\ 1 \end{pmatrix} \cdot \frac{d\underline{s}}{d\theta}\, d\theta$$

$$= \int_0^{2\pi} \begin{pmatrix} -\sin\theta \\ \cos\theta \\ 1 \end{pmatrix} \cdot \begin{pmatrix} -\sin\theta \\ \cos\theta \\ 0 \end{pmatrix} d\theta$$

$$= \int_0^{2\pi} 1 \, d\theta = 2\pi$$

7



(We can't use Stokes to evaluate $\int_\Sigma \underline{F} \cdot d\underline{A}$!)

"Write down the unit normal" $\hat{\underline{n}}(x,y,z) = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$, use $d\underline{A} = \hat{\underline{n}}\, dA$:

$$\text{So } \int_\Sigma \underline{F} \cdot d\underline{A} = \int \begin{pmatrix} y \\ x \\ z \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix} dA = \int_\Sigma (2xy + z^2)\, dA = 2\pi/3$$

First term vanishes by symmetry, second is $\frac{1}{2} \cdot \frac{4\pi}{3}$ by our "$x^2+y^2+z^2$" symmetry trick.

8 Solve first $\frac{\partial \phi}{\partial x} = \sin y - z\cos x$

$$\therefore \quad \phi = x\sin y - z\sin x + c(y,z)$$

then solve $\frac{\partial \phi}{\partial y} = x\cos y + \sin z$

these cancel

$$\text{ie} \quad \frac{\partial}{\partial y}\left(x\cancel{\sin y} - z\cancel{\sin x} + c(y,z)\right) = x\cancel{\cos y} + \sin z$$

$$\frac{\partial c(y,z)}{\partial y} = \sin z$$

$$\therefore \quad c(y,z) = -\cos z + e(z)$$

Now solve $\frac{\partial \phi}{\partial z} = y\cos z - \sin x$

cancel

$$\text{i.e.} \quad \frac{\partial}{\partial z}\left(x\cancel{\sin y} - z\sin x - \cos z + e(z)\right) = y\cos z - \sin x$$

$$\frac{\partial e(z)}{\partial z} = y\cos z - \sin z$$

$$\therefore \quad e(z) = y\sin z + \cos z$$

so $\phi = x\sin y - z\sin x - \cancel{\cos z} + y\sin z + \cancel{\cos z} + d$

$\quad = x\sin y - z\sin x + y\sin z + d$

Solve for constant by plugging in $x = y = z = \frac{\pi}{2}$

$$0 = \phi(\pi/2, \pi/2, \pi/2) = \frac{\pi}{2}\cdot 1 - \frac{\pi}{2}\cdot 1 + \frac{\pi}{2}\cdot 1 + d$$

$$\therefore \quad \underline{d = -\pi/2}$$

so $\underline{\phi = x\sin y - z\sin x + y\sin z - \pi/2}$

(This can also be found by "guess-&-check"!)

20E Winter 2017 Final Exam - solutions

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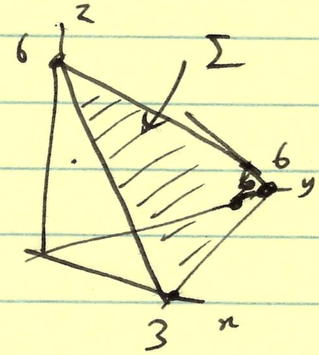
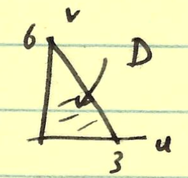
①



Parametrise as a graph, using

$$z = 6 - 2u - v$$

with domain in the (u,v) plane the triangle D as shown $\longrightarrow$



$T_u \times T_v$

$$\text{So} \int_{\Sigma} (x+z)\, dA = \int_0^3 du \int_0^{6-2u} dv\ (u + 6 - 2u - v) . \left\| \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} \right\|$$

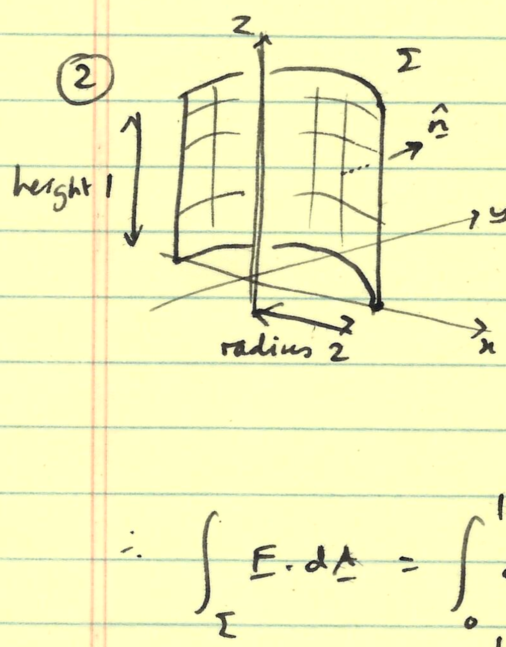
$$= \sqrt{6} . \int_0^3 du \left[-uv + 6v - \frac{v^2}{2} \right]_0^{6-2u}$$

$$= \sqrt{6} . \int_0^3 du \left[-6u + 2u^2 + 36 - 12u - 18 + 12u - 2u^2 \right]$$

$$= \sqrt{6} . \int_0^3 du \left[18 - 6u \right]$$

$$= \sqrt{6} . \left[18u - 3u^2 \right]_0^3 \qquad = \underline{\underline{\sqrt{6} . 27}}$$

②



Use cylindrical coordinates to parametrise:

$0 \le z \le 1$, $\qquad x = 2\cos\theta$

$0 \le \theta \le \pi$ (gives half of cylinder) $\qquad y = 2\sin\theta$

$z = z.$

We know that $\underset{\sim}{T_z} \times \underset{\sim}{T_\theta} = \begin{pmatrix} 2\cos\theta \\ 2\sin\theta \\ 0 \end{pmatrix}$

(standard fact, or work it out!)

$$\therefore \int_{\Sigma} \underline{F} . d\underline{A} = \int_0^1 dz \int_0^{\pi} d\theta \begin{pmatrix} 2\cos\theta \\ 1 \\ z^2 \end{pmatrix} . \begin{pmatrix} 2\cos\theta \\ 2\sin\theta \\ 0 \end{pmatrix}$$

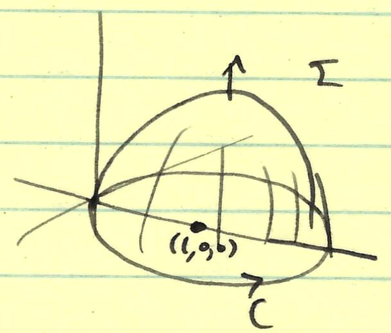
$$= \int_0^1 dz \int_0^{\pi} d\theta \left(4\cos^2\theta + 2\sin\theta \right)$$

$$= 1 \times \int_0^{\pi} d\theta \left(2(1+\cos 2\theta) + 2\sin\theta \right)$$

$$= \left[2\theta + \sin 2\theta - 2\cos\theta \right]_0^{\pi}$$

$$= \underline{\underline{2\pi + 4}}$$

(3)



The hemisphere is centred at (1,0,0) and has radius 1, so its boundary is the circle C parametrised by

$$x = 1+\cos\theta \qquad 0 \le \theta \le 2\pi$$
$$y = \sin\theta$$
$$z = 0$$

Use Stokes' theorem, so

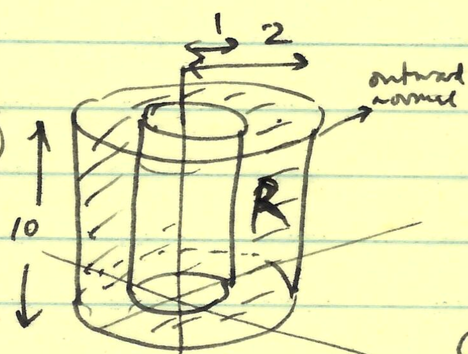
$$\int_{\Sigma} (\underline{\nabla} \times \underline{F}) \cdot d\underline{A} = \int_C \underline{F} \cdot d\underline{s} = \int_0^{2\pi} d\theta \begin{pmatrix} -1.\sin\theta \\ 1+\cos\theta + 0 \\ \cos(1+\cos\theta) \end{pmatrix} \cdot \underbrace{\begin{pmatrix} -\sin\theta \\ \cos\theta \\ 0 \end{pmatrix}}_{\frac{d\underline{s}}{d\theta}}$$

$$= \int_0^{2\pi} d\theta \left(\sin^2\theta + \cos\theta + \cos^2\theta + 0 \right)$$

(fortunately!)

$$= \int_0^{2\pi} d\theta \left(1+\cos\theta \right) = \underline{\underline{2\pi}}$$

(4)



$\Sigma = \partial R$ is a torus-like surface, which ~~but~~ is a bit complicated, so it should be easier to use Gauss' theorem.

Compute $\underline{\nabla} \cdot \underline{F} = 1 + 1 + xy = 2 + xy$

$$\int_{\Sigma = \partial R} \underline{F} \cdot d\underline{A} = \int_R (\underline{\nabla} \cdot \underline{F})\, dV = \int_0^{10} dz \int_0^{2\pi} d\theta \int_1^2 dr \left(2 + r\cos\theta \cdot r\sin\theta \right) r$$

Jacobian factor

[Here I am using ~~the~~ the 3d cylindrical coordinate system $(r, \theta, z) \longmapsto (r\cos\theta, r\sin\theta, z)$ which gives $dx\,dy\,dz = r\,dr\,d\theta \cdot dz$, and that's where the Jacobian factor comes from].

$$\cancel{= 10 \times \int_1^2 dr \left[\ldots \right]}$$

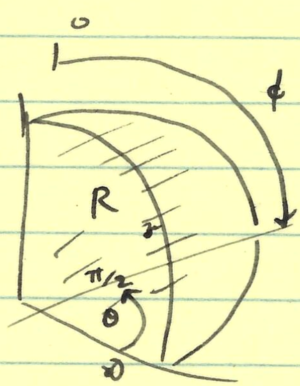
$$= 10 \times \int_0^{2\pi} d\theta \left[r^2 + \frac{r^4}{4} \cos\theta \sin\theta \right]_1^2$$

$$= 10 \times \int_0^{2\pi} d\theta \left(3 + \frac{15}{4} \cdot \frac{\sin 2\theta}{2} \right)$$

$$= 10 \cdot 6\pi = \underline{\underline{60\pi}}$$

(second term of integral vanishes because sine runs over 2 complete cycles.)

(5)



Use 3d spherical coordinates to parametrize the region:

$$0 \le r \le 1$$
$$0 \le \theta \le \pi/2$$
$$0 \le \phi \le \pi/2$$

In these coordinates, the distance of the pt (r, θ, ϕ) from the origin is just "r"! So we need to integrate $\int_R r\,dV$ and then divide by vol(R):

$$\text{Average} = \frac{1}{\text{vol}(R)} \cdot \int_0^{\pi/2} d\theta \int_0^{\pi/2} d\phi \int_0^1 dr \, . \underbrace{r}_{\text{[integrand]}} . \, r^2 \sin\phi$$

[Jacobian, ie. $dV = r^2 \sin\phi \, dr\, d\theta\, d\phi$]

$$= \frac{1}{\frac{1}{8} \cdot \frac{4}{3} \pi . 1^3} \int_0^{\pi/2} d\theta \int_0^{\pi/2} d\phi \left[\frac{r^4}{4} \sin\phi \right]_0^1$$

R is one-eighth of the whole unit ball

$$= \frac{6}{\pi} \cdot \frac{\pi}{2} \times \left[-\frac{1}{4} \cos\phi \right]_0^{\pi/2} = \frac{-3}{4} \cdot \left[0 - 1 \right] = \underline{\underline{\frac{3}{4}}}$$

⑥ We don't know what the curve is, so the only way $\int_\gamma \underline{F}\cdot d\underline{s}$ can be well-defined independent of the knowledge of γ is if $\underline{F}$ is conservative.

So we check the curls:

$$\nabla \times \underline{F} = \begin{pmatrix} \partial/\partial x \\ \partial/\partial y \\ \partial/\partial z \end{pmatrix} \times \begin{pmatrix} 3x^2y^2z \\ 2x^3yz \\ x^3y^2 \end{pmatrix} = \begin{pmatrix} 2yx^3 - 2x^3y \\ 3x^2y^2 - 3x^2y^2 \\ 6x^2yz - 6x^2yz \end{pmatrix} = 0 \quad \therefore \ \underline{F} \text{ is conservative}$$

$$\nabla \times \underline{G} = \begin{pmatrix} \partial/\partial x \\ \partial/\partial y \\ \partial/\partial z \end{pmatrix} \times \begin{pmatrix} 3x^2y^2z \\ 2x^3yz \\ x^2y^3 \end{pmatrix} = \begin{pmatrix} 3y^2x^2 - 2x^3y \\ \sim \\ \sim \end{pmatrix} \neq 0, \quad \therefore \ \underline{G} \text{ is NOT conservative.}$$

So $\int_\gamma \underline{G}\cdot d\underline{s}$ can't be evaluated without knowing γ.

However, $\int_\gamma \underline{F}\cdot d\underline{s} = \phi(1,2,3) - \phi(0,0,0)$,

where ϕ is any potential function satisfying $\nabla\phi = \underline{F}$.

So we must simply find a function ϕ satisfying

$$\frac{\partial \phi}{\partial x} = 3x^2y^2z \qquad \frac{\partial \phi}{\partial y} = 2x^3yz \qquad \frac{\partial \phi}{\partial z} = x^3y^2$$

In fact this is easy to guess & check! Try $\phi = x^3y^2z$ ✓

Hence $\phi(1,2,3) - \phi(0,0,0) = 1^3 . 2^2 . 3 - 0 = 12$

$$\therefore \quad \int_\gamma \underline{F}\cdot d\underline{s} = 12$$

⑦ Parametrise the arc γ as $\underline{s}(t) = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + t \begin{pmatrix} 3 \\ -2 \\ -1 \end{pmatrix}$ $\quad 0 \le t \le 1$

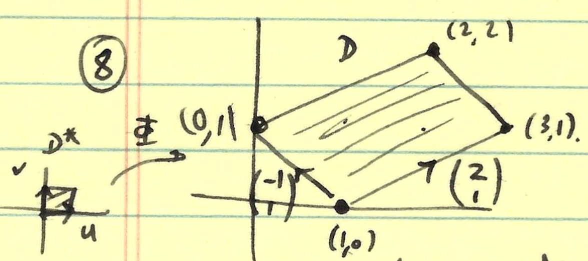
Then
$$\int_\gamma \underline{F}\cdot d\underline{s} = \int_0^1 dt \; \underbrace{\begin{pmatrix} (1+3t)(3-t) \\ 3(2-2t) \\ 1 \end{pmatrix}}_{\underline{F}(\underline{s}(t))} \cdot \underbrace{\begin{pmatrix} 3 \\ -2 \\ -1 \end{pmatrix}}_{\frac{d\underline{s}}{dt}}$$

$$= \int_0^1 dt \left(3(3+8t-3t^2) - 6(2-2t) - 1 \right)$$

$$= \int_0^1 dt \left(-4 + 36t - 9t^2 \right)$$

$$= \left[-4t + 18t^2 - 3t^3 \right]_0^1 = \underline{\underline{11}}$$

8



It's annoying to integrate over D by slicing, so let's reparametrize it to be the unit square D^* in the uv plane by making the transformation

$$\underset{\text{unit square } D^*}{(u,v)} \longmapsto \begin{pmatrix} 1 \\ 0 \end{pmatrix} + u \begin{pmatrix} 2 \\ 1 \end{pmatrix} + v \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

(I'm choosing $(u,v) = (0,0)$ to go to $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$, then the '$\underline{i}$' & '$\underline{j}$' edges of the uv square to go to the $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$ & $\begin{pmatrix} -1 \\ 1 \end{pmatrix}$ sides of the parallelogram.)

So $x = 1 + 2u - v$
$y = 0 + u + v$

$$\Rightarrow \text{Jacobian } \left| \frac{\partial(x,y)}{\partial(u,v)} \right| = \begin{vmatrix} 2 & -1 \\ 1 & 1 \end{vmatrix} = 3.$$

Notice also that $x+y$ becomes $1+3u$ under the transform,

$$\text{so } \int_D \cos(x+y)\, dx\, dy = \int_0^1 du \int_0^1 dv \cdot \cos(1+3u) \cdot 3$$

$$= 3 \times 1 \times \left[\frac{\sin(1+3u)}{3} \right]_0^1$$

$$= \underline{\underline{\sin 4 - \sin 1}}$$